Functions of Bounded Variation on "good" Metric Spaces

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Abstract

In this paper we give a natural definition of Banach space valued BV functions defined on complete metric spaces endowed with a doubling measure (for the sake of semplicity we will say doubling metric spaces) supporting a Poincaré inequality (see Definition 2.5 below). The definition is given starting from Lipschitz functions and taking closure with respect to a suitable convergence; more precisely, we define a total variation functional for every Lipschitz function; then we take the lower semicontinuous envelope with respect to the L^1 topology and define the BV space as the domain of finiteness of the envelope. The main problem of this definition is the proof that the total variation of any BV function is a measure; the techniques used to prove this fact are typical of Γ–convergence and relaxation. In Section 4 we define the sets of finite perimeter, obtaining a Coarea formula and an Isoperimetric inequality. In the last section of this paper we also compare our definition of BV functions with some definitions already existing in particular classes of doubling metric spaces, such as Weighted spaces, Ahlfors–regular spaces and Carnot–Carathéodory spaces.

1 Introduction

In this paper we give a definition of Banach space valued functions with bounded variation on complete metric spaces endowed with a given measure. The metric space is assumed to have some structure and the measure is supposed to satisfy some compatibility conditions with the metric. More precisely, the measure is assumed to be doubling (see Definition 2.1 below); an additional assumption on the metric space X is made, that is a Poincarè inequality is supposed to hold for suitable couples of function, the Lipschitz functions and their "gradients" (see Definition 2.5). We recall that a space with these properties turns out to be almost geodesic, in the sense that it is possible to define a new metric which is geodesic and which is Lipschitz equivalent to the original one (see for example Semmes [33] and Cheeger [8]). Particular examples of doubling metric spaces and Sobolev doubling spaces are given by the weighted Euclidean spaces, i.e. the Euclidean $\mathbb{R}^n$ spaces where the Lebesgue measure is replaced with a suitable absolutely continuous doubling measure (for the main definitions and properties see for example Franchi [14], Franchi–Serapioni–Serra Cassano [15], and Muckenhoupt [31]). Particular examples of doubling spaces are the Ahlfors–regular spaces and among them we recall the Carnot–Carathéodory groups, which are also particular subriemannian manifolds (see for instance Gromov [19] and Nagel–Stein–Wainger [32]).
We recall the definition of BV functions in Euclidean spaces; given an open set $\Omega \subset \mathbb{R}^n$, a function $u \in L^1(\Omega)$ is said to have bounded variation, $u \in \mathrm{BV}(\Omega)$, if

$$\|Du\|(\Omega) := \sup\left\{\int_\Omega u \operatorname{div}\phi dx : \phi \in \mathcal{C}^1_0(\Omega, \mathbb{R}^n), \|\phi\|_\infty \le 1\right\} < +\infty.$$

$\mathrm{BV}(\Omega)$ is a Banach space with the norm

$$\|u\|_{\mathrm{BV}(\Omega)} = \|u\|_{L^1(\Omega)} + \|Du\|(\Omega);$$

the problem of this norm is that smooth functions are not dense in $\mathrm{BV}(\Omega)$. Nevertheless, every function $u \in \mathrm{BV}(\Omega)$ is approximable in a weak sense, see Anzellotti–Giaquinta Theorem in [4];

that is, there exists a sequence $(u_h)_h$ of smooth functions such that

$$\begin{cases} \|u_h - u\|_{L^1} \to 0 \\ \\ \displaystyle\int_\Omega \|\nabla u_h\| dx \to \|Du\|(\Omega). \end{cases}$$

More precisely, we can state the following Proposition.

Proposition 1.1 *Let $u \in L^1(\Omega)$, $\Omega \Subset \mathbb{R}^n$ open; then the following conditions are equivalent:*

1. *$u \in \mathrm{BV}(\Omega)$, i.e.*

$$\|Du\|(\Omega) := \sup \left\{ \int_\Omega u \operatorname{div} \phi dx : \phi \in \mathcal{C}_0^1(\Omega, \mathbb{R}^n), \|\phi\|_\infty \leq 1 \right\} < +\infty;$$

2. *there exists a vector measure $\sigma = (\sigma_1, \ldots, \sigma_n)$ with bounded total variation such that*

$$\int_\Omega u \partial_i \phi dx = - \int_\Omega \phi d\sigma_i, \qquad \forall \phi \in \mathcal{C}_0^1(\Omega)$$

   (that is, the distributional first derivatives of u are measures);

3. *there exist a constant $M > 0$ and a sequence $(u_h)_h \subset \mathcal{C}_0^1(\Omega)$ such that $u_h \to u$ in $L^1(\Omega)$ and*

$$\limsup_{h \to \infty} \int_\Omega \|\nabla u_h\| dx \leq M;$$

4. *there exists a positive finite measure ν in Ω such that*

$$\min_{c \in \mathbb{R}} \int_B \|u - c\| dx \leq r(B) \nu(B),$$

   for every ball $B \subset \Omega$ ($r(B)$ is the radius of B).

Let us observe that statement 3. makes sense in a metric space language, replacing smooth functions with Lipschitz functions. This remark is the starting point of our work. We will use statement 3. to give the definition of BV–functions and then we will see the equivalence between 3. and 4.. We last notice that in the Euclidean case the fact that $\|Du\|$ is a measure follows immediately from property 2.; to prove that 1. implies 2. an important role is played by the Riesz representation Theorem. In a metric setting we cannot use this approach, so in order to prove that $\|Du\|$ is a measure we have to use different techniques (see Theorem 3.4).
The main idea of the definition is then to relax with respect to the topology of $L^1_{loc}(\Omega)$ the functional

$$u \to \int_\Omega \|\nabla u\| d\mu,$$

defined only on Lipschitz functions, and to take as $\mathrm{BV}(\Omega)$ the set where this relaxed functional is finite. We remark that this same method produces the usual Sobolev spaces when relaxing with respect to the L^p topology (see for instance Cheeger [8], Hajłasz–Koskela [23]).
Definitions of BV functions already exist in some particular examples of such metric spaces, such as in the Euclidean weighted spaces (see Baldi [5]), in Finsler geometries (see Bellettini–Bouchitté–Fragalà [6]) and in the Carnot–Carathéodory groups (see for instance Garofalo–Nhieu [17]). At the end of Section 5 we will see that our definition coincides with that one given in the Carnot–Carathéodory case (in this proof the Ahlfors–regularity plays an important role).
The paper is structured as follows; in Section 2 we will give the main definitions and present our setting; in Section 3 we give the definition of BV functions and give the main properties; in Section 4 we define the sets of finite perimeter in the same way as Caccioppoli [7] and De Giorgi [10] did (see also Federer [13], Giusti [18], Mattila [29], Maz'ya [30] and Ziemer [34] as further references); in Section 5 we will give some examples of particular doubling metric spaces and we will see the equivalence with existing definitions of BV–functions.
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2 Basic definitions

We give here the main definitions and the main properties of our ambient space; for this section we refer to Cheeger [8], Hajłasz–Koskela [23] and Heinonen–Koskela–Shanmugalingam–Tyson [26]. Given a complete metric space (X, d), we shall indicate with $B = B_\varrho(x)$ the open ball centered at x with radius $r(B) = \varrho > 0$; by $B' = \lambda B$ we mean the ball with the same center of B and with radius $r(B') = \lambda r(B)$. Moreover, for any subset $\Omega \subseteq X$, we denote by $\Delta(\Omega)$ the diameter of Ω, i.e.

$$\Delta(\Omega) = \sup \{d(x, y) : x, y \in \Omega\} .$$

We shall denote by $\mathfrak{B}$ the family of all balls of X, i.e. $B \in \mathfrak{B}$ if and only if there exist $x \in X$ and $0 < \varrho < \Delta(X)$ such that $B = B_\varrho(x)$; by $\mathcal{T}_X$ we shall denote the collection of all open subsets of X and by $\mathcal{B}(X)$ the σ–algebra of Borel subsets of X. By a metric measure space (X, d, μ) we simply mean a metric space (X, d) endowed with a non–trivial Borel measure μ (i.e. a measure on Borel subsets). We give the following definition.

Definition 2.1 (Doubling Space) *A doubling space is a complete metric measure space (X, d, μ) such that*

$$\mu(2B) \leq c\mu(B) \qquad \text{for every ball } B \in \mathfrak{B} \tag{1}$$

for some constant $c \geq 0$; the best constant c_D satisfying (1) is called the doubling constant for μ. Notice that for a non–trivial doubling measure we have that the support is all the space X.

Remark 2.2 Given a doubling space, it is possible to prove that

$$\frac{\mu(B_r(x))}{\mu(B_R(y))} \geq \frac{1}{c_D^2} \left(\frac{r}{R}\right)^{\log_2 c_D} \tag{2}$$

for every $x, y \in X$ and $R \geq r > 0$ with $x \in B_R(y)$. Then, a metric measure space is doubling if and only if there exist constants $c', s > 0$ such that

$$\frac{\mu(B_r(x))}{\mu(B_R(y))} \geq c' \left(\frac{r}{R}\right)^s$$

for all $x, y \in X$ and for all $R \geq r > 0$ with $x \in B_R(y)$. We notice in particular that every doubling measure space has some kind of dimension, the constant $s = \log_2 c_D$ (called the homogeneous dimension); we will find this dimension in the definition of Sobolev spaces and in the immersion theorems for Sobolev spaces (see Remark 2.8 below).

We recall some important properties of doubling spaces; a basic tool will be played by the following proposition, which gives a partition of unity. For a proof of it see for instance Appendix B of Gromov [20].

Proposition 2.3 (Partition of Unity) *Let $t > 0$ be a fixed number; then there exists a subset $A(t)$ of X such that*

- *$d(a_1, a_2) \geq t$ for all $a_1, a_2 \in A(t)$ with $a_1 \neq a_2$;*
- *$X \subset \bigcup_{a \in A(t)} B_a$, where $B_a = B_t(a)$.*

Moreover, for any $k > 0$ there exists a constant $\beta(k) > 0$ depending only on k and on the doubling constant c_D such that

- *$\sum_{a \in A(t)} \mathbf{1}_{kB_a}(x) \leq \beta(k)$ for every $x \in X$, where $\mathbf{1}_E$ denotes the characteristic function of the set E.*

In addition, we can find a family $\{\phi_a^{(t)}\}_{a \in A(t)}$ of real–valued Lipschitz functions on X such that

- *$0 \leq \phi_a^{(t)} \leq \mathbf{1}_{2B_a}$;*
- *the functions $\phi_a^{(t)}$ have Lipschitz constant not greater than Λ/t, where $\Lambda > 0$ is a constant depending only on the doubling constant c_D;*

- $\sum_{a\in A(t)} \phi_a^{(t)}(x) \equiv 1$ *for all* $x \in X$.

Another easy consequence of the definitions is that any doubling metric space is proper, i.e. closed and bounded sets are compact. In fact it is possible to prove that there exists a constant $c > 0$ depending only on the doubling constant c_D such that every ball of radius R can be covered by at most $c(R/r)^s$ balls of radii r (for a proof, see for instance David–Semmes [9]); this implies that the balls are totally bounded and then relatively compact. Since any open set can be approximated from inside by closed and bounded sets, this enables us to say that a property holds locally on an open set Ω if it holds on every compact set $K \subseteq \Omega$ (or equivalently on every open set $A \Subset \Omega$). For example, if we denote with $\mathrm{Lip}(\Omega; V)$ the set of Lipschitz functions $u : \Omega \to V$ with $(V, \|\cdot\|)$ a Banach space, then the space $\mathrm{Lip}_{loc}(\Omega; V)$ is the set of functions $u : \Omega \to V$ such that $u \in \mathrm{Lip}(A; V)$ for every open set $A \Subset \Omega$. Similarly, if we denote with $L^p(\Omega, \mu; V)$ the space of p–integrable functions on Ω, we can define the space $L^p_{loc}(\Omega, \mu; V)$. We shall write simply $L^p(\Omega, \mu)$ (respectively $L^p_{loc}(\Omega, \mu)$) in the case when $V = \mathbb{R}$. Given a function $u \in L^1_{loc}(\Omega, \mu; V)$, we set

$$u_B = \frac{1}{\mu(B \cap \Omega)} \int_{B\cap\Omega} u(x) d\mu(x).$$

We recall that when we say that a Banach space valued function is integrable, we mean the Bochner integrability (see for example Heinonen–Koskela–Shanmugalingam–Tyson [26]).
Given a curve $\gamma : [0, 1] \to X$, we denote the length of γ by

$$\Lambda(\gamma) = \sup \sum d(\gamma(t_i), \gamma(t_{i-1}))$$

where the supremum in taken over all possible finite partition $[t_{i-1}, t_i]$ of the interval $[0, 1]$. With the notation $\gamma : x \to y$ we mean that $\gamma : [0, 1] \to X$ is a rectifiable curve joining x and y, i.e. γ is a curve with finite length such that $\gamma(0) = x$ and $\gamma(1) = y$.
Given a continuous function $u : X \to V$, an upper gradient for u is a Borel function $g : X \to [0, +\infty]$ such that for every $x, y \in X$ and for every $\gamma : x \to y$ there holds

$$\|u(y) - u(x)\| \le \int_0^1 g(\gamma(s))\|\dot\gamma\|(s) ds,$$

where

$$\|\dot\gamma\|(s) = \limsup_{h\to 0} \frac{\|\gamma(s + h) - \gamma(s)\|}{|h|}. \tag{3}$$

We notice that if γ is a Lipschitz curve the limsup is at almost every point a limit; this limit is called *metric derivative* of γ. We denote by $UG(u)$ the collection of all upper gradients of u.

Remark 2.4 [Properties of upper gradients] If $u_i : X \to V$ are continuous and $g_i \in UG(u_i)$ $(i = 1, 2)$, then

1. $|\alpha_1|g_1 + |\alpha_2|g_2 \in UG(\alpha_1 u_1 + \alpha_2 u_2)$ for every $\alpha_1, \alpha_2 \in \mathbb{R} \setminus \{0\}$;
2. $(\|u_1\| + \varepsilon)g_2 + (\|u_2\| + \varepsilon)g_1 \in UG(u_1 \cdot u_2)$ for every $\varepsilon > 0$.

We give the following definition.

Definition 2.5 (Poincaré Space) *We will say that the measure metric space (X, d, μ) is a Poincaré space if μ is a doubling measure and if, in addition, X supports a weak Poincaré inequality, i.e. for every pair (u, g) with $u \in \mathrm{Lip}_{loc}(X; V)$ and $g \in UG(u)$, there holds*

$$\int_B \|u(x) - u_B\| d\mu(x) \le c_* r(B) \int_{\lambda B} g(x) d\mu(x) \qquad \textit{for any ball } B \subseteq X, \tag{4}$$

for some absolute constants $\lambda \ge 1$, $c_* \ge 0$.

Remark 2.6 Due to Heinonen–Koskela–Shanmugalingam–Tyson [26, Theorem 4.3], the following statemants are equivalent;

1. there exists a Banach space $(V, \|\cdot\|)$ such that (X, d, μ) supports a Poincaré inequality;

2. (X, d, μ) supports a Poincaré inequality for $V = \mathbb{R}$;

3. (X, d, μ) supports a Poincaré inequality for every Banach space V.

In particular it is possible to see that the constant c_* in (4) can be choosen independent of the Banach space V.

We recall that a metric space (X, d, μ) supporting the Poincaré inequality is quasi–convex, i.e. there exists a constant $\alpha > 0$ such that for every $x, y \in X$ there exists a curve $\gamma : x \to y$ with $\Lambda(\gamma) \le \alpha d(x, y)$. In other words, if $\varrho \ge d$ is the geodesic metric on X defined by

$$\varrho(x, y) = \inf \{\Lambda(\gamma) | \gamma : x \to y\},$$

then ϱ is Lipschitz equivalent to d. On the other hand, if X is a proper space (i.e. closed and bounded sets are compact, as in the case of doubling spaces), then for any $x, y \in X$, there exists a curve $\gamma : [0, 1] \to X$ such that

$$\varrho(x, y) = \Lambda(\gamma).$$

We recall the following definition.

Definition 2.7 (Chain Condition) *A subset $\Omega \subset X$ is said to satisfy a (σ, M)–chain condition with σ, $M \ge 1$, if for each point $x \in \Omega$ there exists a sequence of balls $(B_h)_h$ such that*

1. *$\sigma B_h \subseteq \Omega$ for every $h \in \mathbb{N}$;*

2. *there exists $h_0 \in \mathbb{N}$ such that for all $h \ge h_0$ the ball B_h is centered at x;*

3. *if r_h denotes the radius of B_h, then*

$$\frac{1}{M} \frac{\Delta(\Omega)}{2^h} \le r_h \le M \frac{\Delta(\Omega)}{2^h} \qquad \forall h \in \mathbb{N};$$

4. *there exists a ball B'_h contained in $B_h \cap B_{h+1}$ such that $B_h \cup B_{h+1} \subseteq MB'_h$.*

For instance, balls in geodesic metric spaces satisfy a suitable chain–condition. It is a well known result (see for instance Koskela Hajłasz–Koskela [23, 22] and Heinonen [25]) that if the space X supports a weak Poincaré inequality and Ω is an open set which satisfies a (σ, M)–chain condition for some σ, $M \ge 1$, then Ω supports a plain Poincaré inequality, i.e.

$$\int_\Omega \|u(x) - u_\Omega\| d\mu(x) \le C\Delta(\Omega) \int_\Omega g(x) d\mu(x), \qquad \forall u \in \mathrm{Lip}_l oc(\Omega; V)$$

with C depending only on c_D, c_*, σ and M. Then, if X is a Poincarè space and the metric is geodesic the strong Poincarè inequality holds, i.e. there exists a constant $c_P > 0$ such that

(5) $$\int_B \|u(x) - u_B\| d\mu(x) \le c_P r(B) \int_B g(x) d\mu(x) \qquad \text{for any ball } B \in \mathfrak{B}.$$

Given $u \in \mathrm{Lip}_{loc}(X; V)$, we define the modulus of the gradient of u as

$$\|\nabla u\|(x) = \liminf_{\varrho \to 0} \frac{1}{\varrho} \sup_{y \in \overline{B_\varrho(x)}} \|u(y) - u(x)\|.$$

This defines a Borel function which is an upper gradient for u (see for instance Cheeger [8, Proposition 1.11] in the case $V = \mathbb{R}$); we give a sketch of the proof. Let us consider an element $v \in V^*$ in the dual of V with $\|v\| = 1$ and let us define

$$g_v(t) = \langle u(\gamma(t)), v\rangle;$$

this is a real valued Lipschitz function, then it is differentiable at almost every point and

$$g_v(1) - g_v(0) = \int_0^1 g'_v(t) dt.$$

We want to prove that $g_v'(t) \leq \|u(\gamma(t))\| \cdot \|\dot\gamma\|(t)$ at every differentiability point of g_v; let us then consider a differentiability point t of g_v. For every $r > 0$ there exists an $h_r > 0$ such that $\|\gamma(t+h_r) - \gamma(t)\| = r$; notice that $h_r \to 0$ as $r \to 0$. Then, by the Schwarz inequality, we obtain that

$$\begin{aligned}\frac{|g_v(t+h_r) - g_v(t)|}{|h_r|} &\leq \frac{\|u(\gamma(t+h_r)) - u(\gamma(t))\|}{h_r} = \frac{\|u(\gamma(t+h_r)) - u(\gamma(t))\|}{r}\frac{\|\gamma(t+h_r)-\gamma(t)\|}{h_r}\\ &\leq \frac{1}{r}\sup_{y\in\overline{B_r(\gamma(t))}}\|u(y) - u(\gamma(t))\|\frac{\|\gamma(t+h_r)-\gamma(t)\|}{h_r}.\end{aligned}$$

Then, by taking the limit as $r \to 0$, we obtain that

$$|g_v'(t)| \leq \|\nabla u\|(\gamma(t))\|\dot\gamma\|(t);$$

we then obtain that

$$\|u(y) - u(x)\| = \sup_{v\in V^*,\|v\|=1}\langle u(y) - u(x), v\rangle = \sup_{v\in V^*,\|v\|=1} g_v(1) - g_v(0) \leq \int_0^1 \|\nabla u\|(\gamma(t))\|\dot\gamma\|(t)dt.$$

We define the Sobolev space following the definitions of Hajłasz [21], Hajłasz-Koskela [23]; we say that a function $u \in L^p(\Omega,\mu;V)$ $(p \geq 1)$ is a Sobolev function, $u \in W^{1,p}(\Omega,\mu;V)$, if there exists a non–negative function $g \in L^p(\Omega,\mu)$ such that

$$\|u(x) - u(y)\| \leq d(x,y)(g(x) + g(y)), \qquad \forall x,y \in X \setminus N,$$

where N is a subset of X with $\mu(N) = 0$. The space $W^{1,p}_{loc}(\Omega,\mu;V)$ is defined in the obvious way.

Remark 2.8 In this paper we will use the following properties of the $W^{1,p}(X,\mu;V)$ Sobolev spaces;

1. $\mathrm{Lip}_{loc}(X;V) \subseteq W^{1,p}_{loc}(X,\mu;V)$ for every $p \geq 1$;

2. if the homogeneous dimension s satisfies $s > 1$, there exists a constant $c_S > 0$ such that, for any ball $B \subseteq X$

$$\left(\frac{1}{\mu(B)}\int_B \|u(x) - u_B\|^{s^*}d\mu(x)\right)^{1/s^*} \leq c_S r(B)\frac{1}{\mu(B)}\int_{\lambda B}\|\nabla u\|(x)d\mu(x), \tag{6}$$

   for any $u \in \mathrm{Lip}_{loc}(X;V)$, where $s^* = s/(s-1)$ (this implies that $W^{1,1}_{loc}(X,\mu;V)$ embeds in $L^{s^*}_{loc}(X,\mu;V)$ in a continuous way, see Hajłasz–Koskela [23, Theorem 5.1]);

3. for any bounded sequence $(u_h)_h \subseteq W^{1,1}_{loc}(X,\mu;V)$ there exists $q > 1$ such that, up to subsequences, $(u_h)_h$ converges in $L^\omega_{loc}(X,\mu;V)$ norm to a function $u \in L^q_{loc}(X,\mu;V)$ for all $1 \leq \omega < q$ (see Hajłasz–Koskela [23, Theorem 8.1]). The exponent q depends only on the homogeneous dimension s and on the Sobolev immersion (6).

Given $u \in \mathrm{Lip}_{loc}(X)$, we have that $A \to \int_A \|\nabla u\|d\mu$ defines a positive Radon measure on $\mathcal{B}(X)$; this will be the starting point of the theory of BV–functions on a Poincaré space; the idea is in fact to relax the functional $u \to \int \|\nabla u\|d\mu$, obtaining a set functional $\|Du\|$ and taking as $\mathrm{BV}(X;V)$ the domain of finiteness of the relaxed functional $\|Du\|$. We notice that, for $p > 1$, we can define in the same way the Sobolev spaces: it suffices in fact to relax the functional

$$u \to \int \|\nabla u\|^p d\mu$$

with respect to the L^p topology. It is easy to see that this procedure gives the same Sobolev spaces defined by Hajłasz–Koskela [23] (see for instance Cheeger [8]).

3 Definition of BV and some properties

Let $\Omega \subseteq X$ be an open set; since $\mathrm{Lip}_{loc}(\Omega; V)$ is dense in $L^1_{loc}(\Omega, \mu; V)$, it makes sense to define, for every $u \in L^1_{loc}(\Omega, \mu; V)$, the total variation of u on every open set $A \subseteq \Omega$ as

$$(7) \qquad \|Du\|(A) = \inf\left\{\liminf_{h\to\infty} \int_A \|\nabla u_h\|(x)d\mu : (u_h)_h \subseteq \mathrm{Lip}_{loc}(A; V), u_h \stackrel{L^1_{loc}(A,\mu;V)}{\longrightarrow} u\right\}.$$

Definition 3.1 (Functions with bounded total variation) *A function $u \in L^1_{loc}(\Omega, \mu; V)$ is said to have locally bounded total variation on Ω if $\|Du\|(A) < +\infty$ for Severy open subset $A \Subset \Omega$. A function is said to have bounded total variation on Ω if $\|Du\|(\Omega) < +\infty$. The vector space of functions with (locally) bounded total variation will be denoted by $\mathrm{BV}(\Omega, \mu; V)$ ($\mathrm{BV_{loc}}(\Omega, \mu; V)$).*

We notice that the definition of $\mathrm{BV}(\Omega, \mu; V)$ coincides with the Euclidean one modulus the summability, i.e. once one requires in addition that $u \in L^1(\Omega)$. When the open set A has some kind of regularuty, for instance when A is a John domain, then the definition given in (7) is equivalent to the following

$$\|Du\|(A) = \inf\left\{\liminf_{h\to\infty} \int_A \|\nabla u_h\|(x)d\mu : (u_h)_h \subseteq \mathrm{Lip}(A; V), u_h \stackrel{L^1(A,\mu;V)}{\longrightarrow} u\right\}.$$

Moreover, we notice that for any given $u \in \mathrm{BV_{loc}}(\Omega, \mu; V)$ and for any $A \subset \Omega$ open, there exists a sequence $(u_h)_h \in \mathrm{Lip}_{loc}(A; V)$ such that $u_h \to u$ in $L^1_{loc}(A, \mu; V)$ and

$$\|Du\|(A) = \lim_{h\to\infty} \int_A \|\nabla u_h\| d\mu.$$

Remark 3.2 We have the following consequences of the definitions; for every $u, v \in L^1_{loc}(X, \mu; V)$, for every $\alpha \in \mathbb{R}$ and for every $A, B \in \mathcal{T}_X$,

1. $\|D(\alpha u)\|(A) = |\alpha| \cdot \|Du\|(A)$;
2. $\|D(u + v)\|(A) \leq \|Du\|(A) + \|Dv\|(A)$;
3. $\|Du\|(A \cup B) \geq \|Du\|(A) + \|Du\|(B)$ if $A \cap B = \emptyset$;
4. $\|Du\|(A \cup B) = \|Du\|(A) + \|Du\|(B)$ if $\mathrm{dist}(A, B) > 0$.

The first three properties are easy to prove; for the fourth one, it suffices to consider to sequences $(u_h)_h \in \mathrm{Lip}_{loc}(A; V)$, $(v_h)_h \in \mathrm{Lip}_{loc}(B; V)$ converging to u in $L^1_{loc}(A, \mu; V)$ and $L^1_{loc}(B, \mu; V)$ respectively, such that

$$\lim_{h\to\infty} \int_A \|\nabla u_h\| d\mu = \|Du\|(A), \qquad \lim_{h\to\infty} \int_B \|\nabla v_h\| d\mu = \|Du\|(B).$$

Then, if we define

$$w_h = \begin{cases} u_h & \text{on } A \\ v_h & \text{on } B, \end{cases}$$

we get a new sequence still converging to u and, using the fact that A and B are distant sets, such that

$$\|Du\|(A \cup B) \leq \liminf_{h\to\infty} \int_{A\cup B} \|\nabla w_h\| d\mu = \|Du\|(A) + \|Du\|(B).$$

Since the other inequality follows from statement 3., we get equality in 4..

The main problem here is to prove that $\|Du\|$ defines a measure; it is not possible in this context to use the Euclidean techniques such as Riesz representation Theorem. The approach here is typical in the study of relaxations problems.
The following Lemma is useful for the proof of Theorem 3.4: for a proof see Ambrosio–Dal Maso [2] or Ambrosio–Mortola–Tortorelli [3].

Lemma 3.3 *Let $u \in L^1_{loc}(\Omega, \mu; V)$ and let $M, N \in \mathcal{T}_\Omega$.*

1. *If N is bounded and $\partial N \cap \partial M = \emptyset$, then there exist open sets $H \Subset M \cap N$, $C_1, C_2 \subset M \cup N$ and a constant $c = c(M, N)$ such that for every $\varepsilon > 0$ and $u \in \mathrm{Lip}(M; V)$, $v \in \mathrm{Lip}(N; V)$, it is possible to find $w \in \mathrm{Lip}(M \cup N; V)$ such that*

$$\int_{M\cup N} \|\nabla w\| d\mu \leq \int_M \|\nabla u\| d\mu + \int_N \|\nabla v\| d\mu + c(M, N) \int_H \|u - v\| d\mu + \varepsilon; \tag{8}$$

$$w \equiv u \text{ on } M \setminus N, \qquad w \equiv v \text{ on } N \setminus M; \tag{9}$$

$$\int_K \|w - \sigma\| d\mu \leq \int_{K_1} \|u - \sigma\| d\mu + \int_{K_2} \|v - \sigma\| d\mu, \qquad \forall \sigma \in L^1_{loc}(M \cup N, \mu; V), \tag{10}$$

whenever $K \Subset M \cup N$ and $K_1 = K \cap C_1 \Subset M$, $K_2 = K \cap C_2 \Subset N$.

2. *If $M' \Subset M$ and $N' \Subset N$, then there exist an open set $H \Subset M \cap N$ and a constant $c > 0$ depending only on M, N, M' and N' such that for every $\varepsilon > 0$ and $u \in \mathrm{Lip}(M; V)$, $v \in \mathrm{Lip}(N; V)$, it is possible to find $w \in \mathrm{Lip}(M \cup N; V)$ satisfying*

$$\int_{M'\cap N'} \|\nabla w\| d\mu \leq \int_M \|\nabla u\| d\mu + \int_N \|\nabla v\| d\mu + c \int_H \|u - v\| d\mu + \varepsilon. \tag{11}$$

PROOF: Let us prove 1.: since $\overline{N \setminus M}$ and $\overline{M \setminus N}$ are disjoint, it is possible to find a function $\phi \in \mathrm{Lip}\, M \cup N$ such that $0 \leq \phi \leq 1$ and

$$\phi \equiv \begin{cases} 1 & \text{on a neighbourhood of } \overline{N \setminus M} \\ 0 & \text{on a neighbourhood of } \overline{M \setminus N}. \end{cases}$$

We then define

$$C_1 = \{\phi < 1\} \cap (M \cup N), \quad C_2 = \{\phi > 0\} \cap (M \cup N), \quad H = C_1 \cap C_2 \Subset M \cap N.$$

Fixed a real number $\varepsilon > 0$, we can find $k \in \mathbb{N}$ such that

$$\int_H (\|\nabla u\| + \|\nabla v\|)\, d\mu \leq k \cdot \varepsilon.$$

Let $\eta = \mathrm{dist}(\partial N, \partial M) > 0$ and define

$$H_i = \left\{ x \in H : \frac{k+i-1}{3k}\eta < \mathrm{dist}(x, \partial N) \leq \frac{k+i}{3k}\eta \right\}, \qquad i = 1, \ldots, k.$$

We can find functions $\phi_i \in \mathrm{Lip}(M \cap N)$ such that $0 \leq \phi_i \leq 1$, $\|\nabla \phi_i\|_\infty \leq 4k/\eta$ and

$$\phi_i(x) = \begin{cases} 1, & \text{if } \mathrm{dist}(x, \partial N) \leq \dfrac{k+i-1}{3k}\eta \\ 0, & \text{if } \mathrm{dist}(x, \partial N) \geq \dfrac{k+i}{3k}\eta. \end{cases} .$$

We then define $w_i = \phi_i u + (1 - \phi_i) v$ and we obtain that

$$\int_{M\cup N} \|\nabla w_i\| d\mu \leq \int_M \|\nabla u\| d\mu + \int_N \|\nabla v\| d\mu + \int_{H_i} (\|\nabla u\| + \|\nabla v\|)\, d\mu + \frac{4k}{\eta} \int_{H_i} \|u - v\| d\mu.$$

But then, summing over i we have that

$$\frac{1}{k} \sum_{i=1}^{k} \int_{M\cup N} \|\nabla w_i\| d\mu \leq \int_M \|\nabla u\| d\mu + \int_N \|\nabla v\| d\mu + \varepsilon + \frac{4}{\eta} \int_H \|u - v\| d\mu,$$

hence there exists an index $i_0 \in \{1, \ldots, k\}$ such that (8) holds with $w = w_{i_0}$. In order to prove (10), if $K \subset M \cup N$ is any compact set, then

$$\begin{aligned}\int_K \|w_{i_0} - \sigma\| d\mu &\leq \int_{K\cap M\setminus N} \|u-\sigma\| d\mu + \int_{K\cap N\setminus M} \|v-\sigma\| d\mu \\ &\quad + \int_{K\cap M\cap N} (\phi_{i_0}\|u-\sigma\| + (1-\phi_{i_0})\|v-\sigma\|) d\mu \\ &\leq \int_{K_1} \|u-\sigma\| d\mu + \int_{K_2} \|v-\sigma\| d\mu.\end{aligned}$$

Let us now prove 2.; fix $\varepsilon > 0$ and, given $\eta = \text{dist}(M', \partial M) > 0$, we define

$$H = N' \cap \left\{ x \in M : \frac{\eta}{3} < \text{dist}(x, M') < \frac{2\eta}{3}. \right\}$$

Clearly $H \Subset M \cap N$; given u and v, we can find a number $k \in \mathbb{N}$ such that

$$\int_H (\|\nabla u\| + \|\nabla v\|)\, d\mu \leq k\varepsilon. \tag{12}$$

We then define

$$H_i = \left\{ x \in X : \frac{k+i-1}{3k}\eta < \text{dist}(x, M') \leq \frac{k+i}{3k}\eta \right\}, \qquad i = 1, \ldots, k.$$

There exist functions $\phi_i \in \text{Lip}(X)$ such that

$$\|\nabla \phi_i\|_\infty \leq \frac{4k}{\eta}$$

$$\phi_i \equiv \begin{cases} 1, & \text{if } \text{dist}(x, M') \leq \frac{k+i-1}{3k}\eta \\ 0, & \text{if } \text{dist}(x, M') \geq \frac{k+i}{3k}\eta. \end{cases}$$

We then define $w_i = \phi_i u + (1-\phi_i)v$ and arguing as above, it is possible to find an index $i_0 \in \{1, \ldots, k\}$ such that (11) holds with $w = w_{i_0}$.

□

Now we are in the position to prove the following Theorem.

Theorem 3.4 *For any $u \in L^1_{loc}(\Omega, \mu; V)$, the set function $\|Du\|$ is the restriction to the open subsets of X of a positive finite measure in X.*

PROOF: Due to De Giorgi–Letta [11, Theorem 5.1], it suffices to prove that the function $\|Du\|$ defined on the class $\mathcal{T}_X$ of open subsets of X satisfies the following properties:

1. $\|Du\|(B) \leq \|Du\|(A)$ if $B \subseteq A$;
2. $\|Du\|(A \cup B) \geq \|Du\|(A) + \|Du\|(B)$ whenever $A \cap B = \emptyset$;
3. $\|Du\|(A) = \sup\{\|Du\|(B) : B \Subset A\}$;
4. $\|Du\|(A \cup B) \leq \|Du\|(A) + \|Du\|(B)$ for every A, B.

The first two properties are easy consequences of the definitions and of properties of the liminf. Let us prove point 3.; let $A \in \mathcal{T}_X$ be an open set and let $x_0 \in X$ be a fixed point. Let us then define the sets

$$A_j = \left\{ x \in A : \text{dist}(x, \partial A) > \frac{1}{j} \right\} \cap B_j(x_0)$$

and the sequence of open relatively compact sets given by

$$\begin{cases} C_1 = A_2 \\ C_k = A_{2k} \setminus \overline{A}_{2k-3}, \quad \forall k \geq 2. \end{cases}$$

Clearly, we can assume that

$$\sup_{B \Subset A} \|Du\|(B) < +\infty;$$

then, since the two families $\{C_{2k}\}$ and $\{C_{2k+1}\}$ are well separated, we have that for every $\varepsilon > 0$ there exists $\bar{k} \in \mathbb{N}$ such that

$$\sum_{k \geq \bar{k}} \|Du\|(C_k) \leq \frac{\varepsilon}{6}. \tag{13}$$

We put $B = A_{2\bar{k}-2}$. We have the following Claim.

Claim 1 *There exists an open set $B' \Subset B$ and a sequence $(u_h)_h \subset \mathrm{Lip}(A \setminus \bar{B}'; V)$ such that $u_h \to u$ in $L^1_{loc}(A \setminus \bar{B}', \mu; V)$ and*

$$\limsup_{h\to\infty} \int_{A\setminus\bar{B}'} \|\nabla u_h\| d\mu \leq \frac{\varepsilon}{3}. \tag{14}$$

PROOF: We put $B' = A_{2\bar{k}-3}$ and we rename $D_h = C_{\bar{k}+h-1}$; we can then consider a sequence $\psi_{m,h} \in \mathrm{Lip}(D_h; V)$ such that $\psi_{m,h} \to u$ in $L^1_{loc}(D_h, \mu; V)$ as $m \to \infty$ and

$$\int_{D_h} \|\nabla \psi_{m,h}\| d\mu \leq \|Du\|(D_h) + \frac{1}{m2^h}.$$

We want then define a sequence $u_{m,h} \in \mathrm{Lip}(\bigcup_{i=1}^h D_i)$ inductively; we put $u_{m,1} = \psi_{m,1}$, and for $h > 1$ we use Lemma 3.3 with $M = D_{h+1}$, $N = \bigcup_{i=1}^h D_i$, $u = \psi_{m,h+1}$, $v = u_{m,h}$ and $\varepsilon_h = \varepsilon/(12 \cdot 2^h)$. Without loss of generality, we may assume that

$$c_h \int_{H_h} \|\psi_{m,h+1} - \psi_{m,h}\| d\mu \leq \frac{\varepsilon}{12 \cdot 2^h},$$

where c_h and H_h are given by Lemma 3.3 and depend only on h. We then obtain that

$$\begin{aligned} \int_{\bigcup_{i=1}^{h+1} D_i} \|\nabla u_{m_h+1}\| d\mu &\leq \int_{D_{h+1}} \|\nabla \psi_{m,h+1}\| d\mu + \int_{\bigcup_{i=1}^{h} D_i} \|\nabla u_{m_h}\| d\mu \\ &\quad + c_h \int_{H_h} \|\psi_{m,h+1} - u_{m,h}\| d\mu + \frac{\varepsilon}{12 \cdot 2^h}. \end{aligned}$$

Moreover, we have that

$$\begin{cases} u_{m,h+1} \equiv \psi_{m,h+1}, & \text{on } D_{h+1} \setminus D_h \\ u_{m,h+1} \equiv u_{m,h}, & \text{on } \bigcup_{i=1}^h D_i \setminus D_{h+1}, \end{cases}$$

and then by induction we obtain that

$$\begin{aligned} \int_{\bigcup_{i=1}^{h+1} D_i} \|\nabla u_{m,h+1}\| d\mu &\leq \sum_{j=1}^{h+1} \int_{D_j} \|\nabla \psi_{m,j}\| d\mu \\ &\quad + \sum_{j=1}^{h} c_j \int_{H_j} \|\psi_{m,j+1} - \psi_{m,j}\| d\mu + \sum_{j=1}^{h} \frac{\varepsilon}{12 \cdot 2^j}. \end{aligned}$$

We then define the sequence

$$u_m(x) = u_{m,h}(x), \qquad \forall x \in \bigcup_{i=1}^{h-1} D_i;$$

so we get that

$$\begin{aligned} \int_{A\setminus\overline{B}'} \|\nabla u_m\| d\mu &= \lim_{h\to\infty} \int_{\bigcup_{j=1}^{h-1} D_j} \|\nabla u_{m,h}\| d\mu \leq \lim_{h\to\infty} \int_{\bigcup_{j=1}^{h} D_j} \|\nabla u_{m,h}\| d\mu \\ &\leq \sum_{j=1}^{\infty} \int_{D_j} \|\nabla \psi_{m,j}\| d\mu + \sum_{j=1}^{\infty} \frac{\varepsilon}{6 \cdot 2^j} \\ &\leq \sum_{j=1}^{\infty} \|Du\|(D_j) + \frac{1}{m} + \frac{\varepsilon}{6} \leq \frac{1}{m} + \frac{\varepsilon}{3}, \end{aligned}$$

thanks to (13). Then condition (14) is satisfied; it remains to prove the convergence in $L^1_{loc}(A \setminus \overline{B}', \mu; V)$ to u. First of all we prove by induction on h that for every $h \in \mathbb{N}$ we have

$$\lim_{m\to\infty} \|u_{m,h} - u\|_{L^1_{loc}(\bigcup_{i=1}^{h+1} D_i)} = 0. \tag{15}$$

For $h = 1$ we have that $u_{m,1} = \psi_{m,1}$, and then (15) follows by assumption. Let us suppose the thesis for h; then from Lemma 3.3, if $K \Subset \bigcup_{i=1}^{h+1} D_i$, there exist $K' \Subset D_{h+1}$ and $K'' \Subset \bigcup_{i=1}^{h} D_i$ depending only on h such that

$$\int_K \|u_{m,h+1} - u\| d\mu \leq \int_{K'} \|\psi_{m,h+1} - u\| d\mu + \int_{K''} \|u_{m,h} - u\| d\mu,$$

and then we are done by assumption on the $\psi_{m,h}$'s and by the inductive hypothesis. Now we prove that $u_m \to u$ in $L^1_{loc}(A \setminus \overline{B}', \mu; V)$. we take $K \Subset A \setminus \overline{B}'$; then there exists a $h \in \mathbb{N}$ such that $K \Subset \bigcup_{i=1}^{h} D_i$. Since by construction we have that $u_m(x) = u_{m,h+1}(x)$ for every $x \in K$ we obtain that

$$\int_K \|u_m - u\| d\mu = \int_K \|u_{m,h+1} - u\| d\mu \overset{m\to\infty}{\longrightarrow} 0.$$

□

We now consider a sequence $v_h \in \mathrm{Lip}(B; V)$ converging to u in $L^1_{loc}(B, \mu; V)$ and such that

$$\int_B \|\nabla v_h\| d\mu \to \|Du\|(B).$$

Using again Lemma 3.3, we can link this sequence and the sequence of the Claim in a new sequence w_h, an open set $H \Subset B \cap (A \setminus \overline{B}')$ not depending on h in order to get

$$\int_A \|\nabla w_h\| d\mu \leq \int_{A\setminus\overline{B}'} \|\nabla u_h\| d\mu + \int_B \|\nabla v_h\| d\mu + c(B, B') \int_H \|u_h - v_h\| d\mu + \frac{\varepsilon}{3};$$

then we have that

$$\|Du\|(A) \leq \|Du\|(B) + \varepsilon.$$

It remains to prove the sub-additivity; we prove a weak sub-additivity, i.e. if $A, B \in \mathcal{T}_X$, we prove that for every $A' \Subset A$ and $B' \Subset B$ we have

$$\|Du\|(A' \cup B') \leq \|Du\|(A) + \|Du\|(B). \tag{16}$$

The sub-additivity will then follow from the inner regularity by passing to the supremum on the left hand side. Let us then fix $\varepsilon > 0$ and take two functions $u_\varepsilon \in \mathrm{Lip}(A; V)$, $v_\varepsilon \in \mathrm{Lip}(B; V)$ such that

$$\int_A \|\nabla u_\varepsilon\| d\mu \leq \|Du\|(A) + \varepsilon, \qquad \int_B \|\nabla v_\varepsilon\| d\mu \leq \|Du\|(B) + \varepsilon.$$

Using Lemma 3.3, we get a function $w_\varepsilon \in \mathrm{Lip}(A \cup B; V)$, $c_\varepsilon > 0$ and an open set $H_\varepsilon \Subset A \cap B$ depending only on ε such that

$$\int_{A'\cup B'} \|\nabla w_\varepsilon\| d\mu \leq \int_A \|\nabla u_\varepsilon\| d\mu + \int_B \|\nabla v_\varepsilon\| d\mu + c_\varepsilon \int_{H_\varepsilon} \|u_\varepsilon - v_\varepsilon\| d\mu + \varepsilon.$$

Then inequality (16) follows by passing to the limit $\varepsilon \to 0$.

□

Remark 3.5 Every function $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega, \mu; V)$ satisfies the weak Poincaré inequality

$$\int_B \|u(x) - u_B\| d\mu(x) \leq C r(B) \|Du\|(\lambda B) \tag{17}$$

for every ball $B \subseteq \Omega$. This is a straightforward consequence of (4) and of the definition of $\|Du\|(\lambda B)$. The same argument based on inequality (6), gives that

$$\|u - u_B\|_{L^{\frac{s}{s-1}}(B)} \leq c_S \left(\frac{r(B)^s}{\mu(B)}\right)^{1/s} \|Du\|(\lambda B). \tag{18}$$

As we will see in the next Section, this inequality will imply the isoperimetric inequality for sets of finite perimeter. We finally recall that in the geodesics case the constant λ can be taken to be equal to one.

Given a Lipschitz function u, we have defined, for any fixed open set $A \in \mathcal{T}_X$, the quantities $\int_A \|\nabla u\| d\mu$ and $\|Du\|(A)$; at this stage we have only that

$$\|Du\|(A) \leq \int_A \|\nabla u\| d\mu. \tag{19}$$

We are not able either to prove equality, nor to give an example showing that these two quantities can be different. What is possible to say is that they are comparable. Indeed, from (19) we have that $\|Du\| \ll \mu$, so that there exists a function $g_u \in L^1(X, \mu)$ such that

$$\|Du\|(A) = \int_A g_u d\mu.$$

Moreover, if we define the function

$$h(x) = \limsup_{\varrho \to 0} \frac{1}{\varrho} \fint_{B_\varrho(x)} \|u(y) - u_{B_\varrho(x)}\| d\mu(y), \tag{20}$$

it is possible to prove that h is comparable with $\|\nabla u\|$ (when u is Lipschitz), i.e. there exist a constant $c_0 \geq 1$ such that

$$\frac{1}{c_0} \|\nabla u\|(x) \leq h(x) \leq c_0 \|\nabla u\|(x), \quad \widetilde{\forall} x \in X.$$

Then, using the Poincaré inequality, we have

$$\frac{1}{\varrho} \fint_{B_\varrho(x)} \|u - u_{B_\varrho(x)}\| d\mu \leq c_P c_D^2 \lambda^s \fint_{B_{\lambda\varrho}(x)} g_u d\mu,$$

which implies $h(x) \leq c_P c_D^2 \lambda^s g_u(x)$, $\widetilde{\forall} x \in X$, whence

$$\int_A \|\nabla u\| d\mu \leq c_0 c_P c_D^2 \lambda^s \|Du\|(A).$$

Equality in (19) depends on the lower semi-continuity of the functional

$$u \to \int_A \|\nabla u\| d\mu, \qquad A \in \mathcal{T}_X,$$

with respect to the L^1 topology; this happens for instance in the Euclidean case and in the C–C spaces (see Section 5.3) since

$$\int_A \|\nabla u\| d\mu = \sup \left\{ \int_A u \operatorname{div} \phi d\mu : \phi \in \mathcal{A} \right\},$$

where $\mathcal{A}$ is a class of admissible functions (for example the class of smooth compactly supported functions with norm less or equal than one).
The following proposition can be proved by a diagonal argument using the definition of $|Du|(A)$.

Proposition 3.6 (Lower Semi-continuity) *Let $\Omega \subseteq X$ be an open set and let $(u_h)_h$ be a sequence in $\mathrm{BV}_{\mathrm{loc}}(\Omega, \mu; V)$ such that $u_h \to u$ in $L^1_{loc}(\Omega, \mu; V)$; then*

$$\|Du\|(A) \leq \liminf_h \|Du_h\|(A), \qquad \text{for any open set } A \subseteq \Omega.$$

In particular, if $\sup_h \|Du_h\|(A) < +\infty$ for any open set $A \Subset \Omega$, the limit function u is in $\mathrm{BV}_{\mathrm{loc}}(\Omega, \mu; V)$.

The following theorem easily follows by the compactness of the embedding of $W^{1,1}_{loc}(\Omega; V)$ in $L^1_{loc}(\Omega; V)$.

Theorem 3.7 (Compactness) *Let $(u_h)_h \subseteq \mathrm{BV}_{\mathrm{loc}}(\Omega, \mu; V)$ be a bounded sequence with respect to the norm of $L^1_{loc}(\Omega, \mu; V)$ and satisfying $\sup_h \|Du_h\|(A) < +\infty$ for any open set $A \Subset \Omega$. Then there exist $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega, \mu; V)$ and a subsequence $(u_{h_k})_k$ converging to u in $L^1_{loc}(\Omega, \mu; V)$.*

We end this section with the following Theorem, which gives the equivalence of point 3. 4. of Proposition 1.1.

Theorem 3.8 *Let (X,d,μ) be a Poincaré space, $\Omega \subset X$ open and let $u \in L^1(X,\mu;V)$; then the following conditions are equivalent;*

- *$u \in \mathrm{BV}(\Omega,\mu;V)$;*
- *there exists a positive finite measure ν such that for every ball $B = B_\varrho(x)$ with $\lambda B \subset \Omega$ there holds*

(21) $$\min_{c\in\mathbb{R}} \int_B \|u-u_B\| d\mu \le \varrho \nu(\lambda B);$$

moreover there exists a constant $c = c(c_D)$ such that $\|Du\| \le c\nu$.

PROOF: Let us consider the sets $A(1/h)$ and the partitions of unity given by Proposition 2.3. We define the sequence of functions

$$u_h(x) = \sum_{a\in A(1/h)} u_{B_a}\phi_a^{(h)}(x).$$

It is clear that $u_h \in \mathrm{Lip}(X;V)$ (it is locally a finite combination of Lipschitz functions). We first prove that $u_h \to u$ in L^1; let us denote by $\hat c_a$ the optimal constant for (21) on $2B_a$. We have that

$$u(x) - u_h(x) = \sum_{a\in A(1/h)} (u(x) - u_{B_a}))\phi_a^{(h)}(x),$$

then using the fact that $\phi_a^{(h)} \le \mathbf{1}_{2B_a}$, we obtain that

$$\begin{aligned}\int_X \|u-u_h\| d\mu &\le \sum_{a\in A(1/h)} \int_{2B_a} \|u-u_{2B_a}\| d\mu \\ &\le \sum_{a\in A(1/h)} \left(\int_{2B_a} \|u-\hat c_a\| d\mu + \|\hat c_a - u_{B_a}\|\mu(2B_a)\right).\end{aligned}$$

But

$$\|\hat c_a - u_{B_a}\| \le \frac{1}{\mu(B_a)}\int_{B_a}\|u-\hat c_a\|d\mu \le \frac{1}{\mu(B_a)}\int_{2B_a}\|u-\hat c_a\|d\mu;$$

then

$$\begin{aligned}\int_X \|u-u_h\| d\mu &\le \sum_{a\in A(1/h)} \left(\frac{2}{h}\nu(2\lambda B_a) + \frac{2}{h}\frac{\mu(2B_a)}{\mu(B_a)\nu(2\lambda B_a)}\right) \\ &\le \frac{4}{h}c_D \sum_{a\in A(1/h)} \nu(2\lambda B_a) \le \frac{4}{h}c_D\beta\nu(X).\end{aligned}$$

Now we prove the equiboundness in $\mathrm{BV}(X)$; on every ball B_a we have

$$\int_{B_a}\|\nabla u\|d\mu = \int_{B_a}\|\nabla(u-c_a)\|d\mu \le \sum_{b\frown a}\|c_b - c_a\|\int_X\|\nabla\phi_b^{(h)}\|d\mu \le \sum_{b\frown a}\|c_b-c_a\|Ch\mu(2B_b).$$

But, if $b\sim a$, then we have that

$$\|c_b - c_a\| \le \frac{c}{h}\frac{\nu(\hat B_a)}{\mu(B_a)},$$

and then

$$\int_{B_a}\|\nabla u_h\|d\mu \le \sum_{b\frown a}\frac{c}{h}\nu(\hat B_a)\nu(\hat B_a)ch\mu(2B_b) \le c\beta\nu(\hat B_a).$$

In conclusion we find that

$$\int_X \|\nabla u_h\|d\mu \le \beta \sum_{a\in A(1/h)} \|\nabla u_h\|d\mu \le c\beta \sum_{a\in A(1/h)} \nu(\hat B_a)$$

and so the theorem is proved.

□

4 Sets of finite perimeter

In this section we define the sets of finite perimeter; the idea of the definition is the same of that of Euclidean case, that is the definitions given by Caccioppoli [7] and De Giorgi [10], (see also Federer [13], Giusti [18], Mattila [29] Maz'ya [30] and Ziemer [34]). From now on we will assume that $V = \mathbb{R}$, and then we shall denote by $|\cdot|$ the Euclidean norm on $\mathbb{R}$.

Definition 4.1 (Caccioppoli Sets) *Let $\Omega \subseteq X$ be an open set and let $E \in \mathcal{B}X$. We say that E has (locally) finite perimeter in Ω if $\mathbf{1}_E \in \mathrm{BV}(\Omega,\mu)$ ($\mathrm{BV}_{\mathrm{loc}}(\Omega,\mu)$). In the sequel we will indicate with* $\mathrm{Cacc}(\Omega)$ ($\mathrm{Cacc}_{\mathrm{loc}}(\Omega)$) *the class of the sets of (locally) finite perimeter in Ω and we will write $\|\partial E\|(A)$ in place of $\|D\mathbf{1}_E\|(A)$ for any Borel set $A \subseteq \Omega$.*

A possible question at this stage is the existence of Caccioppoli sets. We have the following Proposition.

Proposition 4.2 (Coarea Formula) *For any $u \in L^1_{loc}(\Omega,\mu)$; if we set $E_t = \{u > t\}$, we have*

$$\int_{-\infty}^{+\infty} \|\partial E_t\|(A)dt = \|Du\|(A). \tag{22}$$

for every open set $A \in \mathcal{T}_\Omega$. In particular, if $u \in \mathrm{BV}(X,\mu)$, then for almost every $t \in R$ the set E_t has finite perimeter and formula (22) holds for every Borel set $A \in \mathcal{B}(X)$.

PROOF: Following Evans [12], given $u \in \mathrm{Lip}_{loc}(\Omega)$ and $A \Subset \Omega$, we define the function

$$m(t) = \int_{E_t \cap A} \|\nabla u\|(x)d\mu(x), \tag{23}$$

where $E_t = \{u > t\}$. The function m is a non–decreasing and bounded, hence differentiable at almost every t. Let then t be a differentiability point of m and define the functions $(g_h)_h : \mathbb{R} \to \mathbb{R}$

$$g_h(s) = \begin{cases} 1 & s \le t \\ h(t-s)+1 & t < s \le t+1/h \\ 0 & s > t+1/h. \end{cases} \tag{24}$$

We define the sequence $v_h(x) = g_h(u(x))$; in this way we obtain that $v_h \to \mathbf{1}_{E_t}$ in $L^1(A,\mu)$. Indeed

$$\begin{aligned} \int_A |v_h(x) - \mathbf{1}_{E_t}(x)|d\mu(x) &= \int_{\{t<u\le t+1/h\}\cap A} g_h(u(x))d\mu(x) \\ &\le \mu(\{t < u \le t+1/h\}) \to 0, \end{aligned}$$

because $\{t < u < t+1/h\} \searrow \emptyset$. So it suffices to prove that the quantities $|Dv_h|(X)$ are bounded. For this purpose, we note that

$$\begin{aligned} \int_A |\nabla v_h(x)|d\mu(x) &\le h\int_{\{t<u\le t+1/h\}\cap A} |\nabla u|(x)d\mu(x) \\ &= h(m(t+1/h) - m(t)). \end{aligned} \tag{25}$$

Then, passing to the limit $h \to \infty$ in (25), we have that

$$\|\partial E_t\|(A) \le \limsup_h \|Dv_h\|(A) \le m'(t). \tag{26}$$

Integrating (26), we get

$$\int_{-\infty}^{+\infty} \|\partial E_t\|(A)dt \le \int_A |\nabla u|d\mu. \tag{27}$$

By approximation and using the lower semi-continuity of the perimeter, we obtain the same inequality for every BV function u (see Evans-Gariepy [12] for details). For the reverse inequality, assuming that u takes values in $[-1,1]$, for any fixed $h \in \mathbb{N}$ we consider numbers $t_{j,h} \in ((j-1)/h - 1, j/h - 1)$ $(j = 1,\ldots,2h)$ such that

$$\frac{1}{h}\|\partial E_{j,h}\|(A) \le \int_{\frac{j-1}{h}-1}^{\frac{j}{h}-1} \|\partial E_t\|(A)dt,$$

where $E_{j,h} = \{u > t_{j,h}\}$. Then we define the sequence

$$u_h(x) = -1 + \frac{1}{h}\sum_{j=1}^{2h} \mathbf{1}_{E_{j,h}}(x).$$

It is clear that

$$\|Du_h\|(A) \le \int_0^1 \|\partial E_t\|(A)dt. \tag{28}$$

Then we are done if we prove that $u_h \to u$ in $L^1(A)$. We define the sets

$$F_{i,h} = \{t_{i,h} < u \le t_{i+1,h}\};$$

then

$$E_{j,h} = \bigcup_{i=j}^{2h} F_{i,h},$$

and

$$u_h(x) = -1 + \frac{1}{h}\sum_{j=1}^{2h}\sum_{i=j}^{2h} \mathbf{1}_{F_{ih}}(x) = -1 + \frac{1}{h}\sum_{i=1}^{2h} i\mathbf{1}_{F_{ih}}(x).$$

In this way we obtain that, since on $F_{i,h}$ we have that $|u - i/h + 1| \le 1/h$,

$$\int_A |u - u_h| d\mu = \sum_{i=1}^{2h}\int_{F_{i,h}} \left|u - \frac{i}{h} + 1\right| d\mu \le \frac{1}{h}\mu(A) \to 0.$$

because $A \Subset \Omega$. Clearly a little modification of this proof gives (28) if u takes values on $[-n, n]$ for any $n \in \mathbb{N}$; then (22) follows by taking the limit as $n \to \infty$.

□

Remark 4.3 We notice that if $u \in \mathrm{BV}(X, \mu)$, then we obtain the following general Coarea Formula

$$\int_{-\infty}^{+\infty}\left(\int_A v(x)d\|\partial E_t\|(x)\right)dt = \int_A v(x)d\|Du\|(x),$$

for any measurable function $v : X \to \mathbb{R}$ and $A \in \mathcal{B}(X)$.

This lemma enables us to obtain a large class of Caccioppoli sets, that is almost all sub-level set of Lipschitz functions. In particular, by considering the distance function from the center, we have that almost every ball has finite perimeter.

Corollary 4.4 *Let $x_0 \in X$ be fixed; then for almost every $\varrho > 0$ the ball $B_\varrho(x_0)$ has finite perimeter.*

The following theorem gives a local version of the isoperimetric inequality for sets of finite perimeter.

Theorem 4.5 (Isoperimetric Inequality) *Let X be a Poincaré space; then there exists a constant $c_I > 0$ such that*

$$\min\{\mu(E \cap B), \mu(E^c \cap B)\} \le (2c_S)^{\frac{s}{s-1}}\left(\frac{\varrho^s}{\mu(B)}\right)^{\frac{1}{s-1}}\|\partial E\|(\lambda B)^{\frac{s}{s-1}} \tag{29}$$

for every $E \in \mathrm{Cacc}(X)$ and for every $B = B_\varrho(x) \subseteq X$.

PROOF: Let us suppose that $\mu(E \cap B) \le \mu(E^c \cap B)$; then if $u = \mathbf{1}_E$,

$$\begin{aligned}\|u - u_B\|_{L^{\frac{s}{s-1}}(B)} &\ge \mu(E \cap B)^{\frac{s-1}{s}}\frac{\mu(E^c \cap B)}{\mu(B)} \\ &\ge \frac{1}{2}\min\{\mu(E \cap B), \mu(E^c \cap B)\}^{\frac{s-1}{s}}.\end{aligned}$$

Then, by (18) we obtain that

$$\min\{\mu(E\cap B),\mu(E^c\cap B)\}^{\frac{s-1}{s}} \le 2c_S\left(\frac{\varrho^s}{\mu(B)}\right)^{\frac{1}{s}}\|\partial E\|(\lambda B),$$

which proves the theorem.

□

Remark 4.6 Inequality (29) reduces, in the Ahlfors–regular case (see Definition 36), to

$$\min\{\mu(E\cap B_\varrho(x)),\mu(E^c\cap B_\varrho(x))\} \le c_I\|\partial E\|(B_{\lambda\varrho}(x))^{\frac{s}{s-1}}, \tag{30}$$

where $c_I=(2c_S)^{s/(s-1)}c_1^{1-s}$.

We end this section with the following Proposition.

Proposition 4.7 *The class* Cacc(X) *is an algebra, i.e. we have that*

1. $\emptyset, X\in \mathrm{Cacc}(X)$;
2. *if* $E\in\mathrm{Cacc}(X)$, *then* $E^c\in\mathrm{Cacc}(X)$ *and* $\|\partial E^c\|=\|\partial E\|$;
3. *if* $E_1,E_2\in\mathrm{Cacc}(X)$, *then* $E_1\cap E_2, E_1\cup E_2\in\mathrm{Cacc}(X)$ *and*

$$\|\partial(E_1\cap E_2)\|+\|\partial(E_1\cup E_2)\|\le\|\partial E_1\|+\|\partial E_2\|.$$

PROOF: The first property is trivial. We prove 2.; given $E\in\mathrm{Cacc}(X)$, we take a sequence $(u_h)_h\subseteq\mathrm{Lip}(X)$ converging to $\mathbf{1}_E$ with equibounded total variations. Then the sequence $1-u_h$ converges to $\mathbf{1}_{E^c}$ and

$$\|\partial E^c\|(X)\le\liminf_h|Du_h|(X). \tag{31}$$

We note that (31) gives $\|\partial E^c\|(X)\le\|\partial E\|(X)$; but then, interchanging E^c and E, we obtain that

$$\|\partial E^c\|=\|\partial E\|. \tag{32}$$

To show 3., given $E_1,E_2\in\mathrm{Cacc}(X)$, we have to prove that $E_1\cap E_2, E_1\cup E_2\in\mathrm{Cacc}(X)$. We take $0\le u_h,v_h\le 1$ two sequences of Lipschitz functions with $u_h\to\mathbf{1}_{E_1}$, $v_h\to\mathbf{1}_{E_2}$ and $|Du_h|(X)\to\|\partial E_1\|$, $|Dv_h|(X)\to\|\partial E_2\|$. Then $u_hv_h\to\mathbf{1}_{E_1\cap E_2}$ and $u_h+v_h-u_hv_h\to\mathbf{1}_{E_1\cup E_2}$; on the other hand

$$|\nabla(u_hv_h)|\le(v_h+\varepsilon)|\nabla u_h|+(u_h+\varepsilon)|\nabla v_h|,$$

$$\begin{aligned}|\nabla(u_h+v_h-u_hv_h)| &= |\nabla(u_h+v_h-u_hv_h-1)|=|\nabla(1-u_h)(1-v_h)|\\ &\le (1-u_h)|\nabla v_h|+(1-v_h)|\nabla u_h|.\end{aligned}$$

Putting together these last two equations, we get

$$\begin{aligned}\|\partial(E_1\cup E_2)\|+\|\partial(E_1\cap E_2)\| &\le \int_X|\nabla(u_h+v_h-u_hv_h)|d\mu\\ &\quad+\int_X|\nabla(u_hv_h)|d\mu\\ &\le (1+\varepsilon)\left(\int_X|\nabla u_h|d\mu+\int_X|\nabla v_h|d\mu\right),\end{aligned} \tag{33}$$

which gives the desired inequality.

□

5 Some examples and applications

In this section we will see some particular examples of doubling spaces; more precisely, we will first see the case of Ahlfors–regular metric spaces, and then we will take a little view on the Carnot–Carathéodory spaces (C-C spaces). In the last paragraph we will see the case of weighted Euclidean spaces.

5.1 Weighted Spaces

We compare here the definition of BV–functions given above with that given by Baldi [5]. We have a given weight function $\omega \in A_1^*$, i.e. ω is a lower semi-continuous function such that there exists a positive constant $A > 0$ with

$$\frac{1}{|B|}\int_B \omega dx \leq A\|\omega\|_{L^\infty(B)},$$

for every ball B (here $|B|$ denotes the Lebesgue measure of B). We recall that a function $u \in L^1(\Omega,\omega)$ with Ω an open subset of $\mathbb{R}^n$, is said to be in $BV_\omega(\Omega)$ if

$$var_\omega u(\Omega) := \sup\left\{\int_\Omega u \operatorname{div}\phi dx : |\phi| \leq \omega, \phi \in \mathrm{Lip}_0(\Omega)\right\}.$$

Then, since for $u \in \mathrm{Lip}(\Omega)$, we have that

$$var_\omega u(\Omega) = \int_\Omega |\nabla u|\omega dx$$

and the function $u \mapsto var_\omega u(\Omega)$ is lower semi-continuous, we get that

$$\|Du\|(\Omega) \geq var_\omega u(\Omega) \tag{34}$$

and then $BV_\omega(\Omega) \subseteq \mathrm{BV}(\Omega,\mu)$. On the other hand, if $\omega \in \mathrm{Lip}(\Omega)$, then, for every $u \in BV_\omega(\Omega)$, it is possible to find a sequence $(u_h)_h$ of Lipschitz functions with $u_h \to u$ in L^1 and

$$var_\omega u_h(\Omega) \to var_\omega u(\Omega). \tag{35}$$

Then $\|Du\|(\Omega) \leq var_\omega u(\Omega)$ and then $\mathrm{BV}(\Omega,\mu) \subseteq BV_\omega(\Omega)$; moreover, by (34) and (35), we get $\|Du\|(\Omega) = var_\omega u(\Omega)$. If the weight is not Lipschitz, we can only say that there exists e sequence u_h of Lipschitz functions converging to u such that $var_\omega u_h(\Omega) \leq c \cdot var_\omega u(\Omega)$ for some constant $c > 0$; in this case we have again that the spaces $\mathrm{BV}(\Omega,\mu)$ and $BV_\omega(\Omega)$ coincide, but for the total variations we have only that

$$var_\omega u(\Omega) \leq \|Du\|(\Omega) \leq c var_\omega u(\Omega).$$

5.2 Ahlfors–regular spaces

First of all we recall that an Ahlfors–regular space of dimension $s > 0$ is a measure metric space (X,d,μ) such that there exist two constant $c_2 \geq c_1 > 0$ with

$$c_1 r^s \leq \mu(B_r(x)) \leq c_2 r^s, \qquad \forall x \in X, 0 < r < \Delta(X). \tag{36}$$

In other words, an Ahlfors–regular metric space X is a metric space with a given measure μ which is equivalent up to a multiplicative constant to the s–dimensional Hausdorff measure.
Notice that $(\mathbb{R}^n, |\cdot|, \mathcal{L}^n)$ is Ahlfors–regular with $c_1 = c_2 = \omega_n$; other examples of Ahlfors regular spaces are the Carnot–Carathéodory spaces (see below). Laakso [28] gave examples of s–Ahlfors regular spaces supporting a Poincarè inequality for every real number $s > 1$; Hanson–Heinonen [24] give examples of n–dimensional Ahlfors–regular spaces which support the Poincaré inequality and without manifold points.
We have seen in Proposition 4.2 that the fact that the balls in a metric space have finite perimeter is related with the differentiability property of the function

$$m(t) = \mu(B_t(x)),$$

and the perimeter is controlled by

$$\|\partial B_t\| \leq m'(t).$$

For an Ahlfors–regular space, the control in (36) with exponent s on the measure μ does not imply the same good control with exponent $(s-1)$ on the perimeter measure. Nevertheless, if we denote by

$$E_c = \{t \in [a,b] : \|\partial B_t\| > ct^{s-1}\},$$

then we obtain that

$$|E_c| \equiv \frac{1}{c}.$$

More precisely, we have that

$$\begin{aligned} |E_c| &= \int_{E_c} dt < \int_a^b \frac{\|\partial B_t\|}{ct^{s-1}} dt \leq \int_a^b \frac{m'(t)}{ct^{s-1}} dt \leq \int_a^b \frac{1}{ct^{s-1}} dDm(t) \\ &= \left.\frac{m(t)}{ct^{s-1}}\right|_a^b + \frac{1}{c}\int_a^b (s-1)\frac{m(t)}{t^s} dt \\ &\leq \frac{1}{c}\left(c_2 b - c_1 a + c_2(s-1)(b-a)\right). \end{aligned}$$

which is the desired inequality.
In this context, Ambrosio [1] in a recent paper proved that for any set E of finite perimeter it is always true that $\|\partial E\| \ll \mathcal{H}^{s-1}$. Moreover, it is possible to prove that there exist a constant $c > 0$ and a Borel function $\theta : X \to [c, \infty)$ such that

$$\|\partial E\|(B) = \int_{B \cap \partial^* E} \theta d\mathcal{H}^{s-1}, \qquad \forall B \in \mathcal{B}(X),$$

where $\partial^* E$ is the essential boundary of E, i.e. the set of points where the volume density of E is neither 0 nor 1.

Remark 5.1 If in addition the measure μ is s–uniform, i.e. if there exists a constant $c > 0$ such that

$$\mu(B_r(x)) = cr^s, \qquad \forall x \in X, 0 < r < \Delta(X),$$

then the function $m(t)$ is differentiable at every point. In particular, every ball has finite perimeter and the estimate

$$\|\partial B_t\| \leq cst^{s-1}, \tag{37}$$

for every $x \in X$, $0 < t < \Delta(X)$. Examples of metric measure spaces with uniform measure are given by the Carnot groups, as we will see in subsection 5.3.

We notice that in (37) equality does not hold in general; this happens for example if the metric is geodesics and if in (19) equality holds.

5.3 C-C spaces

A Carnot–Carathéodory $\mathfrak{C}$ space is defined by giving an open set $\Omega \subset \mathbb{R}^n$ and $m < n$ vector fields $X = (X_1, \ldots, X_m)$ with locally Lipschitz coefficients on Ω; a distance on Ω is defined by taking the infimum of length of admissible curves γ joining points in Ω, where an admissible curve is a curve with tangent vector to γ generated at every point by the vector fields X. We shall assume that this distance is everywhere finite; this assumption is ensured if for instance the vector fields are smooth and satisfy the Hörmander condition, that is if the Lie algebra generated by X is the whole space $\mathbb{R}^n$. The measure that we will consider is the Lebesgue measure on $\mathbb{R}^n$.
In this setting the space of bounded variation functions $BV_X(\Omega)$ is usually defined as the space of functions $u \in L^1(\Omega)$ such that

$$\|Du\|_X(\Omega) = \sup\left\{\int_\Omega u \operatorname{div}_{X^*} \phi dx : \phi \in \mathcal{C}_0^1(\Omega, \mathbb{R}^n), \|\phi\|_\infty \leq 1\right\} < +\infty,$$

where $\mathrm{div}_{X^*}\phi = X_i^*\phi^i$, with X^* the adjoint vector field of X. We notice that if u is for example Lipschitz, then

$$\begin{aligned}\int_\Omega \|\nabla u\| dx &= \sup\left\{\int_\Omega \langle \nabla_X u, \phi\rangle dx : \phi \in \mathcal{C}_0^1(\Omega, \mathbb{R}^m), \|\phi\|_\infty \le 1\right\} \\ &= \sup\left\{\int_\Omega u \,\mathrm{div}_{X^*}\phi dx : \phi \in \mathcal{C}_0^1(\Omega, \mathbb{R}^m), \|\phi\|_\infty \le 1\right\} = \|Du\|_X(\Omega),\end{aligned}$$

where by $\nabla_X u$ we mean the vector $(X_1 u, \ldots, X_m u)$. Then, if we take $u \in BV_X(\Omega)$, we can find a sequence of regular functions $(u_h)_h$ converging to u with $\|Du\|_X(\Omega) \to \|Du\|_X(\Omega)$; but then $\|Du\|(\Omega) \le \|Du\|_X(\Omega)$. The reverse inequlity follows by the lower semicontinuity of $\|Du\|_X$.

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