An area formula in metric spaces

Valentino Magnani
Scuola Normale Superiore
Piazza dei Cavalieri 7, 56126 Pisa, Italy
e-mail v.magnani@sns.it

CVGMT Preprint Server - http://cvgmt.sns.it/papers/mag02a/

Abstract

We present an area formula for Lipschitz maps between metric spaces. All the known area formulae for Riemannian manifolds and stratified groups are re-obtained in a unified picture. By means of the area formula for Lipschitz maps between stratified groups we prove a rigidity theorem and we characterize purely unrectifiable stratified groups.

Introduction

The aim of this paper is to present the area formula under minimal conditions in a purely metric setting and to present some new applications when the metric spaces are stratified groups. The metric formulation has the advantage to present a unified approach to area formula in several different contexts, as Euclidean spaces, Riemannian manifolds and stratified groups. The last ones, also called "Carnot groups", constitute important examples of non-Riemannian geometries.

The only available notion of jacobian in metric spaces is essentially obtained in a tautological way, defining the "metric jacobian" as the density of the pull-back measure induced by the Lipschitz map, (Definition 1.4). This notion was already considered in [21] for the case of stratified groups. With this definition our metric area formula amounts to a careful application of differentation theorems on measures. Thus, the metric formulation does not require in principle any differentiability theorem for the Lipschitz map, although we will see that all known cases require these theorems in order to show that the general hypotheses for the metric area formula are satisfied.

The interest to develop tools of classical Analysis in metric spaces is receiving much attention in these last few years. Several recent papers have been done in this perspective. We mention only a few works, [1], [3], [4], [6], [7], [12], [15], [22], without being exhaustive. Some motivations and reasons for this study come from the Analysis in Carnot-Carathéodory spaces, a field that has reached a rapid expansion in different areas, as PDE's, Geometric Measure Theory and Differential Geometry.