

Advertising Intensity and Welfare in an Equilibrium Search Model

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Abstract

We analyze an equilibrium search model in a duopoly setting with bilateral heterogeneities in production and search costs in which firms can advertise by announcing price and location. We study existence, stability, and comparative statics in such a setting, compare the duopolistic advertising level to the socially optimal level, and find conditions in which a duopolist advertises more or less than the social optimum.

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1 Introduction

Imperfect price information is a fundamental aspect of consumer search and creates an ideal structure within which to study advertising where consumers can refine their knowledge of individual firm prices as they receive advertisements. In this paper, we study an equilibrium search model under a duopoly and introduce an advertising technology in which firms can inform some proportion of consumers of their price. Our underlying market structure is similar to that of Bénabou (1993) and Carlson and McAfee (1983) with bilateral heterogeneities in production and search costs. Since we study a duopoly where consumers *ex ante* know the distribution of prices but not each firm's individual price, any consumer receiving at least one advertisement is then perfectly informed of prices. The market consists of a continuum of consumers with individual search costs distributed along the unit interval, similar to Rob (1985), where all consumers enter the market with a free initial search and can choose to visit the other firm at some cost.¹ We ask, given that consumers engage in optimal search, will duopolies tend to over- or under-advertise relative to a planner? Our analysis provides good insight on the interaction between search and advertising in a duopoly setting and enhances our understanding of the welfare effects of advertising with search.

A priori, it is not clear whether the duopolistic advertising level generally exceeds that of a planner or vice versa. The planner realizes the social gains of decreased production and search costs. As such, the planner's decision is based heavily on price dispersion and its subsequent effect on consumer search. But acting purely as a profit maximizer, the only benefit of advertising from the firm's perspective is the additional profit from attracting buyers that might have only purchased from another firm. If we take as our measure of welfare the sum of

¹Our model is therefore a simplified version of Robert and Stahl (1993), Bénabou (1993), and Rob (1985). We assume the first search is free for simplicity and to avoid having to keep track of those consumers who elect not to buy. See Janssen, Moraga-González, and Wildenbeest (2005) for a relaxation of this assumption.

consumer and producer surplus as well as search, advertising, and production costs, and consider inelastic demand so that the sum of consumer surplus and total revenue are fixed, then welfare depends completely on advertising, production, and search costs. In this case, the planner sends buyers to the low price firm only if the decrease in search and production costs exceeds the cost of advertising, a tradeoff which the firm does not consider. The firm and planner therefore have potentially conflicting incentives.

Note that our goal is not to establish equilibrium price dispersion under minimal conditions. Indeed, with bilateral heterogeneities, price dispersion is more or less an automatic byproduct of the assumed market structure. For a clear exposition on more general price dispersed equilibria, see Reinganum (1979), Rob (1985), Burdett and Judd (1983), and Robert and Stahl (1993), among others. Our goal in this paper is to develop an equilibrium search model that highlights the fundamental role of price advertising, and in doing so, provide definitive welfare results.

Given our duopoly setting, the model is fairly general. We allow for a relatively general search cost distribution, potentially downward-sloping demand, and a general advertising function that can accommodate economies of scale in advertising. Under fairly mild assumptions, we prove existence in pure strategies. We then derive comparative statics with respect to the constant costs of production and advertising as well as exogenous changes in advertising intensity. Although these results can go either way, we show that the relevant dynamic stability conditions rule out counter-intuitive comparative statics.

We then turn to welfare issues, the main focus of the paper. The welfare standard we adopt is that of a social planner maximizing welfare, as previously discussed, subject to the duopoly first order conditions for price. We impose the latter constraint because the first best solution of a planner allowed to choose both prices and advertising intensity would be to essentially set the low cost firm's price to zero, making a useful comparison between the market and planner's ad-

vertising level impossible. In other words, the pricing constraint puts the planner on the same footing as the market with respect to the socially optimal choice of advertising.

Our analysis provides intuitive sufficient conditions under which duopolistic advertising intensity exceeds or lies below that of a planner. In general, we find that firms over-advertise when profit margins are sufficiently different or the indifferent consumer's search cost is sufficiently low. We express this result in terms of both production and advertising costs as well as the consumer's maximum willingness to pay, and we provide another sufficient condition with regard to the effectiveness of advertising.² All conditions relate to the tradeoff between advertising and search as a means to disseminate information to consumers. As a general intuition, duopolies exploit the informational role of advertising more than would a planner and are more hesitant to rely on consumer search when margins are largely different and vice versa when margins are relatively close. Under-advertising therefore results when advertising costs are high and the indifferent consumer has a relatively high search cost. In this case, advertising has a clear social benefit as a larger measure of consumers could avoid paying their search costs, but firms do not take appropriate advantage due to the high cost of advertising. For symmetric search cost distributions, an equivalent interpretation is that firms over-advertise when the majority of consumers do not search and under-advertise otherwise.

Previous advertising and sequential search models include Robert and Stahl (1993), Janssen and Non (2005), Butters (1977), and Stegeman (1991). In Robert and Stahl, a finite number of homogeneous firms sell identical goods to a finite number of homogeneous consumers, where firms can inform buyers of their price through costly advertising. Consumers therefore collect information both by receiving advertisements and by sequential search. In such a setting, they show there

²In our model, the form of the advertising function or the search cost distribution determines advertising effectiveness.

exists a unique equilibrium with price dispersion and derive comparative statics with regard to entry, search costs, and advertising costs. While their analysis thoroughly describes the strategic interaction of advertising and search and exhibits equilibrium price dispersion in a sequential search model without *ex ante* heterogeneities, they do not compare the competitive and socially optimal advertising levels.

Janssen and Non (2005) develop a similar model for the special case of a duopoly. Their model differs from Robert and Stahl (1993) primarily in that there is some small percentage of completely informed consumers, i.e., shoppers, which has important implications for comparative statics and especially the limiting cases of zero search or advertising costs. They derive partly contrasting results to Robert and Stahl, exemplifying the critical role of search in consumer and firm behavior with advertising, but they also do not compare the competitive and socially optimal advertising levels.

In Butters' (1977) model, there are an infinite number of consumers and heterogeneous firms producing identical products, where all buyers value the product equally and have homogeneous search costs.³ The advertising technology is such that any given advertisement reaches exactly one consumer, which excludes economies of scale in advertising. Buyers do not adopt an optimal search process due to "certain unpalatable conclusions." Instead, the probability of a given buyer reaching a particular firm is proportional to the firm's sales. As such, his welfare results are only based on optimal *firm* behavior. Moreover, the proportion of sales is endogenous, which is problematic for both firm strategies and the planner's problem. Finally, Butters does not impose the monopolistically competitive pricing constraint on the social planner's problem, making advertising comparisons problematic. Nonetheless, he finds that the competitive advertising level always exceeds that of a planner.

³We discuss here only the version of the model with consumer search. In the initial model without search, Butters finds that firms advertise optimally.

Stegeman (1991) develops a model similar to Butters (1977), although he allows heterogeneous reservation prices so that buyers value the product differently. He derives equivalent results only if search costs are sufficiently small, but generally finds that monopolistically competitive firms advertise *too little*.

Even without search, welfare results are not obvious. Dixit and Norman (1978) show that advertising is excessive, while Shapiro (1980) extends this work, showing that advertising is sometimes under-utilized.⁴ Shapiro, however, only considers the monopoly case, and as Bagwell (2001) shows, Shapiro's model can be extended for several firms, which results in excessive advertising. Without search, the intuition is that advertising usually increases output, which is generally welfare improving. But it might also increase prices and margins, which is welfare reducing.

The remainder of the paper is organized as follows. In Section 2, we develop the model, prove existence and stability of equilibria, and derive comparative statics. Section 3 establishes the major welfare results and characterizes conditions in which the advertising firm over- or under-advertises. Proofs of all Propositions are deferred to the Appendix.

2 The Model

2.1 Model Setup

Consider a search model where consumers are identical except for their search costs. The market is normalized to one, and buyers are identified by their search cost $s \in [0, 1]$. The distribution of search costs follows a cdf $Q(s)$, with pdf $q(s)$ and full support on $[0, 1]$. We assume that $Q(0) = 0$, $q(s)$ is C^2 , and that $q(s)$, $q'(s)$, and $q''(s)$ are bounded with $q_{max} = \sup_{s \in [0,1]} q(s)$, $q_{min} = \inf_{s \in [0,1]} q(s)$,

⁴Butters (1976) and Bagwell (2001) provide good surveys of generally accepted results of the literature and of the ambiguous nature of advertising and its effect on equilibrium outcomes. As Butters describes, a fundamental reason for such ambiguities hinges on the differing views of advertising's effect on sales, which is easily resolved in a search setting.

and similarly for $q'(s)$ and $q''(s)$. We also make the standard assumption that consumers are perfectly informed as to the distribution of prices but are uncertain about which firms offer which price, as in Bénabou (1993) and Carlson and McAfee (1983). Given prices, individual demand arises from a quasi-linear utility function, with indirect utility $v(p) + y$. By Roy's Identity, each consumer purchases $d(p) = -v'(p)$ units at price p . Let p_{max} denote the consumer's maximum willingness to pay, and assume $d(p)$ is C^2 with $d' < 0$ on $[0, p_{max})$. Buyers enter the market via a free first search but must pay their search cost to visit another firm.

In a duopoly without advertising, half of the consumers randomly visit the high cost firm and half visit the low. Of the unlucky buyers reaching the high cost firm, only those consumers with sufficiently low search costs benefit from an additional search. The decision of such a consumer is based on

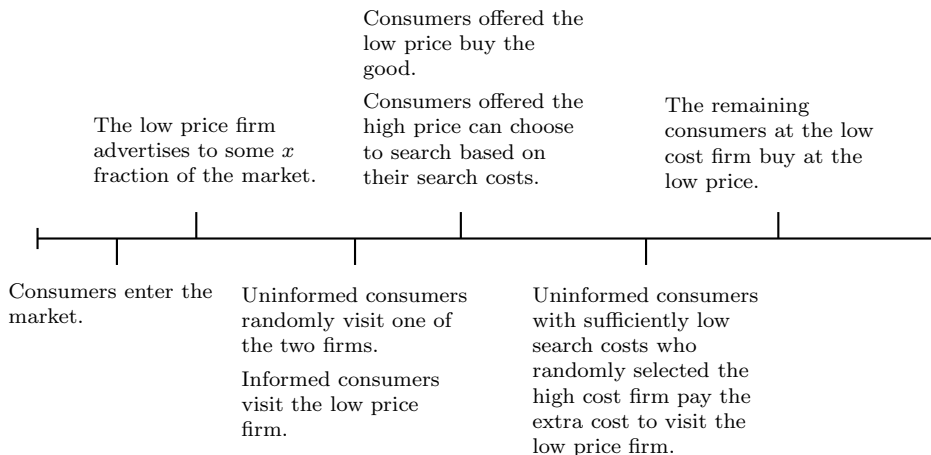
$$v(p_L) - s \geq v(p_H). \quad (2.1)$$

This yields the critical search value $\hat{s} = v(p_L) - v(p_H)$, which is the cost below which consumers search again to find the low cost firm and above which consumers purchase from whichever firm they randomly choose (i.e., for $s \geq \hat{s}$, consumers are inactive searchers). We refer to the buyer with $s = \hat{s}$ as the indifferent consumer.

There are two firms, each producing identical goods with heterogenous costs of production. The low cost firm has marginal cost normalized to zero, while the high cost firm has constant marginal and average costs of $c > 0$. Both firms can advertise their price and location to a fraction of the market at some constant marginal cost $A > 0$. Note that, since the distribution of prices is known, any consumer receiving an advertisement is then perfectly informed of prices, in which case the high price firm never advertises. We denote the level of advertising by $x \in [0, 1]$, where the advertising firm is bound to charge the price advertised (e.g., for legal reasons). Given x , denote the proportion of uninformed consumers by $f(x)$, where $f(x)$ satisfies $f' < 0$, $f'' \geq 0$, $f(0) = 1$, and $f(1) = 0$. Therefore, given

x , the proportion of informed consumers is $1 - f(x)$ drawn uniformly from $[0, 1]$, where each consumer is equally likely to observe an advertisement. Both firms take as given consumer behavior described above, and play the subsequent Nash game with prices and advertising as strategic variables. Figure 1 summarizes the setup thus far.

Figure 1: CONSUMER AND FIRM INTERACTION



Although we formally address this issue in Section 2.2, assume for now that each firm prices according to cost so that the low cost firm is the low price firm. In this case, the low and high price firms face the following demands:

$$q_L = d(p_L) \left[1 - \frac{1}{2} f(x)(1 - Q(\hat{s})) \right]; \quad (2.2)$$

$$q_H = \frac{1}{2} d(p_H) f(x)(1 - Q(\hat{s})). \quad (2.3)$$

In words, (2.3) comes from some proportion—determined by x —of consumers being informed of the low price via advertising, leaving $f(x)$ uninformed. Of these, half randomly select the high cost firm. Finally, some portion $Q(\hat{s})$ have sufficiently low search costs so that they never pay the high price, i.e., they are active searchers. (2.3) is therefore the probability that any given buyer purchases

from the high price firm, where each buyer demands $d(p_H)$ units. The remaining consumers pay the low price and demand $d(p_L)$ units each, which yields (2.2).

2.2 Existence, Stability, and Comparative Statics

We assume the monopolist's problem

$$\max_{p \in [0, p_{max})} \Pi = d(p)(p - c)$$

has a unique solution, denoted p^* , where $\Pi_p > 0$ on $[0, p^*)$. Consumers receive sufficient indirect utility so that they always purchase at the monopoly price (i.e. $v(p^*) + y \geq p^*$). We also assume

$$\Pi_p p > -d_p \Pi, \tag{2.4}$$

which essentially restricts the elasticity of demand. For notational convenience, denote the monopolist's first order condition evaluated at the high cost firm's price and cost by Π_p^H and similarly for the low cost firm.

In this paper, we want to focus on the natural equilibrium where the low cost firm is the low price firm. We therefore begin with an artificially restricted case where the low cost firm must price below the high cost firm. As such, only the low cost firm advertises and faces the profit maximization problem

$$\max_{\substack{p_L \leq p_H \\ x \leq 1}} p_L q_L - Ax, \tag{2.5}$$

which yields the following first order conditions for price and advertising respec-

tively:

$$\frac{\partial \pi_L}{\partial p_L} = \left[1 - \frac{1}{2}f(x)(1 - Q(\hat{s})) \right] \Pi_p^L - \frac{1}{2}q(\hat{s})f(x)d(p_L)^2 p_L = 0; \quad (2.6)$$

$$\frac{\partial \pi_L}{\partial x} = -\frac{1}{2}f'(x)(1 - Q(\hat{s}))d(p_L)p_L - A = 0. \quad (2.7)$$

Similarly, we restrict the high cost firm to price above the low cost firm. The high cost firm therefore does not advertise and must solve

$$\max_{\substack{c \leq p_H \\ p_L \leq p_H < p_{max}}} q_H (p_H - c), \quad (2.8)$$

which yields

$$\frac{\partial \pi_H}{\partial p_H} = \frac{1}{2}f(x)(1 - Q(\hat{s}))\Pi_p^H - \frac{1}{2}f(x)q(\hat{s})d(p_H)^2(p_H - c) = 0. \quad (2.9)$$

Definition 1 *In a restricted game, the low and high cost firms solve (2.5) and (2.8) respectively. We define a **restricted Nash equilibrium** by the triplet (p_L^*, p_H^*, x^*) such that (p_L^*, p_H^*, x^*) is a Nash equilibrium of this restricted game.*

Having defined a restricted Nash equilibrium, we now show that such an equilibrium exists. We then find conditions such that restricted Nash equilibria and conventional Nash equilibria coincide.

Proposition 1 *Given our assumptions on $f(\cdot)$, $q(\cdot)$, and the monopolist's problem, the profit functions, π_L and π_H , are quasi-concave in firms' own actions, strictly concave in p_L and p_H (for a given x), and a restricted pure-strategy Nash equilibrium exists provided the hazard function satisfies*

$$\frac{q'(\hat{s})}{q(\hat{s})} \in \left(-\frac{\Pi_p^H}{\Pi^H d}, \frac{\Pi_p^L}{\Pi^L d} \right). \quad (2.10)$$

Condition (2.10) is a standard hazard condition that imposes restrictions on the

tails of the density. Certainly, the uniform distribution fits this requirement, but in general, any standard hill or bell-shaped density with $q'(s)$ relatively flat in the tails will suffice.

We previously restricted strategy sets in our consideration of the restricted game and restricted Nash equilibrium. We now provide conditions such that the above pricing restrictions are non-binding and the restricted Nash equilibrium is a Nash equilibrium in the conventional sense.

Proposition 2

- (i) *For all $c > 0$, every restricted Nash equilibrium is a Nash equilibrium.*
- (ii) *There exists some $\bar{c} > 0$ such that, for all $c > \bar{c}$, every Nash equilibrium involves the low cost firm pricing below the high cost firm.*

Part (i) simply says that both firms are content with pricing at $p_L^* \leq p_H^*$, while part (ii) ensures that, even if allowed to choose any price up to p_{max} , firms still choose prices consistent with the restrictions of equations (2.5) and (2.8).

We therefore have a duopoly game with heterogeneous consumers and firms in which the low cost firm prices below the high cost firm and can advertise to some fraction of consumers, where uninformed buyers can search for the lowest available price. A price dispersed equilibrium exists and consumers follow an optimal search rule, based on (2.1), so that both advertising and search effectively disseminate information between buyers and firms.

Given the existence of equilibrium pricing and advertising decisions, we now impose stability via a standard proportional marginal profitability adjustment rule.⁵

Proposition 3 *Assuming conditions in Propositions 1 and 2 are satisfied, then*

⁵See equation (C.1) in Appendix C.

given $q(\cdot)$, $d(\cdot)$, Π , and c such that $Q(c) < 1$, the conditions

$$\frac{f'(x)}{f(x)} > -\frac{(1 - Q(c))}{(2p_{max} - c)^2 q_{max}}, \text{ and}$$

$$\frac{f''(x)}{f'(x)} < -\frac{3q_{max}}{1 - Q(c)},$$

are sufficient such that the triplet (p_L^*, p_H^*, x^*) , where $p_L^*, p_H^* \in [0, p_{max})$ and $x^* \in [0, 1]$, is a locally stable Nash equilibrium.

Proposition 3 formalizes the role of the advertising function in determining stability, where we see that $f'(x)$ must be small relative to $f(x)$ and $f''(x)$ large relative to $f'(x)$, in absolute value. Intuitively, this is a standard contraction condition to ensure the effect of any given strategic variable on the marginal profitability of that variable exceeds the effect on the marginal profitability of all other variables.

More specifically, the condition on $f'(x)/f(x)$ roughly states that the normalized rate of informing consumers of prices via advertising must not exceed the rate at which uninformed consumers become inactive searchers due to price changes. If this were not the case, there would be a clear incentive for the low price firm to advertise intensely and effectively ignore the prospect of consumer search as small price changes would have a larger effect on the marginal profitability of advertising rather than price. Similarly, the condition on $f''(x)/f'(x)$ essentially states that the second order effect of advertising (in terms of consumers becoming more informed) must exceed the hazard rate of the search cost distribution. If this does not hold and the second order effect is small, then small changes in x have a larger effect on the marginal profitability of price changes rather than advertising, and the firm would rather rely on consumer search.

Given existence and stability, we can now discuss comparative statics of the game above.

Proposition 4 *Assuming stability as in Proposition 3 and given $q(\cdot)$, $d(\cdot)$, Π and conditions in Propositions 1 and 2, the following relationships hold in equilibrium:*

- (i) p_L is increasing in c and decreasing in A ;
- (ii) p_H is increasing in c and decreasing in A ;
- (iii) x is decreasing in A and is non-monotonic in c , where there exists some c^* such that x is increasing in c for all $c < c^*$ and decreasing in c for all $c > c^*$;
- (iv) price dispersion $p_H - p_L$ is increasing in both c and A ; and
- (v) an exogenous increase in x increases both p_H and p_L and decreases price dispersion.

These results are fairly intuitive. Consider first the response to an increase in the cost of production, c . Naturally, the high price firm must increase price. But since price dispersion partially dictates consumer behavior, the low price firm is now guaranteed a larger proportion of the market via search, which makes the advertising decision slightly more difficult. If c is low, the firm increases advertising intensity, while if c is high, advertising intensity decreases. Intuitively, the low price firm faces a trade-off between choosing advertising intensity and price. If equilibrium prices are such that the marginal benefit of advertising is high relative to that of consumer search, the low price firm is sufficiently responsive to an increase in c so as to accommodate both an increase in price and an increase in x . For small c , the intuition is that the role of search in forming firm demands is relatively small. It makes sense, then, that the low price firm would rather focus on advertising intensity. If cost asymmetries are high, however, the low price firm leans toward a “price-oriented” strategy versus an “advertising intensive” strategy as search plays a much larger role in forming firm demands. As expected, the direct effect of c on p_H is larger than the effect on p_L , so price dispersion is increasing in c .

Now consider the response to an increase in the cost of advertising, A . As expected, the low price firm decreases advertising intensity.⁶ More interestingly, we see that both p_H and p_L also decrease. This result is intuitively similar to

⁶It can be shown that, without stability, counter-intuitive comparative statics might result in which advertising intensity increases with the cost of advertising. See Chapter 4 of Vives (1999) for a thorough explanation.

that above as the low price firm relies on a “price-oriented” strategy. In doing so, the high price firm must respond with a price decrease so as to avoid losing a large market share due to high price dispersion and subsequent consumer search. We therefore find that, for an increase in A and perhaps c , the strategic value of advertising decreases, and firms become more price oriented. Such behavior highlights the important role of search in firm pricing decisions. Again, since A has a direct effect on p_L , price dispersion is increasing in A .

Condition (v) shows that, while individual prices are increasing with advertising intensity, price dispersion is decreasing. An exogenous increase in x therefore has a larger impact on the advertising firm than on the high price firm. This decrease in price dispersion subsequently decreases the proportion of consumers who engage in search. As we will see in Section 3, this tradeoff between advertising and search intensity has important welfare implications.

3 Welfare and Advertising Intensity

We now have a model in which advertising plays a purely informational role in announcing the true price and location of the low cost firm, and thus implicitly doing so for the high cost firm. But as mentioned in Section 1, welfare effects are unclear due to the inherent tension between the social planner and the advertising firm. The planner knows it is socially optimal that consumers reach the low cost firm on their first attempt, rather than pay additional search costs, but does not account for firm profits. The low cost firm, however, cares only about profit and is indifferent to whatever search costs its customers accrue. Our goal now, therefore, is to fully characterize when and how this tension might lead the firm to over- or under-advertising relative to a planner.

Formally, we consider the basic pricing/advertising game proposed in Section 2 and study the duopolistic advertising level relative to the level chosen by a social planner. For simplification, we assume all consumers inelastically demand one

unit up to some maximum price, which fixes consumer surplus and total revenue as a sum so that welfare depends totally on the transaction prices of advertising, production, and search costs. Note that, although we consider inelastic demand, the price level is still relevant as it determines the potential gains from search and therefore affects the value of advertising.

First note that, from Proposition 1, π_L and π_H are strictly concave in p_L and p_H , respectively. Therefore the first order conditions for price are necessary and sufficient for the constrained planner's problem. Denote the welfare attributed to the low and high cost firms by

$$w_L = \bar{u} \left[1 - \frac{1}{2}f(x)(1 - Q(\hat{s})) \right] - \frac{1}{2}f(x) \int_0^{\hat{s}} sq(s)ds - Ax, \text{ and} \quad (3.1)$$

$$w_H = \frac{1}{2}(\bar{u} - c) f(x) (1 - Q(\hat{s})) \quad (3.2)$$

respectively. In words, (3.1) comes from $[1 - \frac{1}{2}f(x)(1 - Q(\hat{s}))]$ consumers receiving utility \bar{u} from purchasing the good, which the firm produces at zero cost. Further, $\frac{1}{2}f(x) \int_0^{\hat{s}} sq(s)ds$ represents those buyers who did not randomly select the low cost firm and who were not informed through advertising but who have sufficiently low search costs so that they pay to visit the other firm. This is a welfare loss as it is the accumulated cost paid by all consumers who search to reach the low price firm. The remaining term, Ax , is the cost of advertising, which decreases welfare by lessening producer surplus. Equation (3.2) is similar and differs due to no advertising, no extra search costs, and positive marginal costs of production.

We can now formally discuss the planner's problem and study existence. Assuming an interior solution, the social planner solves

$$\max_{x \in [0,1]} \bar{u} - Ax - \frac{1}{2}f(x) \int_0^{\hat{s}} sq(s)ds - \frac{1}{2}f(x)(1 - Q(\hat{s}))c, \quad (3.3)$$

subject to

$$\frac{\partial \pi_L}{\partial p_L} = \left[1 - \frac{1}{2}f(x)(1 - Q(\hat{s})) \right] - \frac{1}{2}f(x)q(\hat{s})p_L = 0 \quad (3.4)$$

$$\frac{\partial \pi_H}{\partial p_H} = -q(\hat{s})(p_H - c) + 1 - Q(\hat{s}) = 0, \quad (3.5)$$

where $\hat{s} = p_H - p_L$.

By imposing the duopoly first order conditions, we are in essence focusing on a structural second best where the planner chooses advertising at prices consistent with firm behavior.⁷ We then solve the constraints implicitly for $\hat{s}(x)$ and plug this into equation (3.3).

To ensure uniqueness, we first restrict $q''(\hat{s})$ so that it is not “too” positive, which ensures that the welfare function, after substituting $\hat{s}(x)$, is concave in x . A sufficient condition for this is

$$\frac{q''(\hat{s})}{q(\hat{s})} \leq \frac{-2f''(x)}{f'(x)(2p_{max} - c)c} - \frac{2}{(p_{max} - c)c}, \quad (3.6)$$

which states that, when $q'(\hat{s})/q(\hat{s})$ is highly negative so that there is a large incentive to decrease price, the second order effect is sufficiently small relative to the second order effect of advertising. Intuitively, we need the decreasing sections of the search cost distribution to not decline rapidly. Otherwise, price dispersion plays an overpowering role in the welfare function as even a small increase in advertising forces a large portion of consumers to become inactive searchers. If $Q(\cdot)$ behaves in such a way, then a unique optimal advertising level may not exist because the incentives of the model are too extreme. Condition (3.6) avoids this scenario and ensures that a unique optimal advertising level exists, as given by the following Proposition.

Proposition 5 *Given $f(\cdot)$, A , c , p_{max} , and $Q(\cdot)$ such that conditions (2.10) and (3.6) are satisfied, there exists a unique $\bar{x} \in [0, 1]$ such that \bar{x} maximizes (3.3),*

⁷See Vives (1999) Chapter 6 for a similar approach with product differentiation.

subject to the duopoly first order conditions for price.

We therefore determine over- or under-advertising by imposing the first order condition for advertising from the low price firm, equation (2.7), on the planner's first order condition for equation (3.3). Given strict concavity, the resulting sign indicates whether firms advertise excessively or vice versa. Specifically, denote the planner's objective function by $W(x)$ and the low price firm's first order condition for advertising by $g(x; p_L, p_H)$, then over-advertising results for $\frac{dW(x)}{dx}|_{g(x;\cdot)=0} < 0$ and under-advertising otherwise. Conditions for each result are summarized in Proposition 6.

Proposition 6 *Denote the mean search cost consumer by μ , then given stability and conditions (2.10) and (3.6),*

- (i) *there exists some \bar{c} , \underline{A} , and p_{max} such that the duopolistic advertising level always exceeds the socially optimal level for all $c \geq \bar{c}$, all $A \leq \underline{A}$, or all $p_{max} \leq \frac{2}{q_{max}} - \mu$;*
- (ii) *for the specific advertising function denoted $f(t, x)$, where $f_t < 0$ and $f_x(0)$ sufficiently large, there exists some \bar{t} such that the duopolistic advertising level always exceeds the socially optimal level for all $t \geq \bar{t}$; and*
- (iii) *there exists some cost combination (\bar{A}, \underline{c}) such that, for all $A \geq \bar{A}$ and $c \leq \underline{c}$, the duopolistic advertising level is always below that of a planner, provided $\frac{1-Q(c)}{q(c)} < c$.*

Corollary 1 *There exists some combination of $Q(\cdot)$, $f(\cdot)$, c , and A such that the duopolistic advertising level and the socially optimal level coincide.*

In discussing Proposition 6, we focus first on the incentives facing the firm and then discuss the planner. For the firm, part (i) provides conditions on profit margin differences such that over-advertising results. Specifically, we know from

Proposition 4 that price dispersion is increasing in c at a slower rate than c and that advertising intensity and prices are decreasing in A . Small A or large c therefore imply that profit margins are sufficiently different, i.e., $p_H - c - p_L$ is sufficiently large. This describes a situation in which the low cost firm has significant leverage over its rival firm.

Large margin differences also imply that \hat{s} is small relative to c so that the indifferent consumer has a relatively low search cost. Since the firm only uses advertising to attract high search cost buyers, it makes sense that with more high search cost consumers in the market, the low cost firm over-advertises. This is strongly due to the impact of advertising on consumer behavior. In essence, buyers respond heavily to advertising because they do not care about price dispersion and instead only care about finding a price lower than what they might otherwise pay. In a duopoly, this implies consumers do not care how low p_L is compared to p_H , just so long as $p_L < p_H$. This intuition also holds in a more general context. For instance, with more firms so that price dispersion perhaps arises out of mixed strategies over a particular support, an analogous interpretation is that consumers do not care how wide the support may be, just so long as advertised prices are below the expected price prior to search.

Note that, due to the non-monotonic nature of x with regard to changes in c , costs of production need not be excessively high in order to satisfy “sufficiently large.” As seen in Appendix F, over-advertising occurs when either $(\hat{s} - c)$ is highly negative or when the duopolistic advertising level is already high so that $f(x)$ is small. Due to the non-monotonicity in c , advertising intensity is increasing in c when $(\hat{s} - c)$ is close to zero and decreasing in c when $(\hat{s} - c)$ is highly negative. Therefore, when one condition is not satisfied, the other is more likely so that over-advertising results for a larger range of c .

Now consider the condition on p_{max} . If p_{max} is sufficiently low, the difference in margins as well as the degree of price dispersion itself is irrelevant. The marginal social benefit of a lower price always exceeds the firm’s marginal benefit of ad-

vertising. This is not an unnecessarily strong restriction as μ is always less than one and could be much smaller. Intuitively, over-advertising results here because price dispersion is naturally very low, implying that almost any strategy forcing a decrease in price dispersion is excessive.

Part (i) therefore describes two different scenarios in which duopolies over-advertise: one in which profit margins are sufficiently different so that the low price firm has more leverage, and another in which price dispersion is more or less restrained by the maximum willingness to pay. In the former, advertising has a clear social benefit but is overused by firms, while in the latter, advertising is far less beneficial. But we are also interested in how the overall shape of the advertising function might affect welfare.

Part (ii) formally describes an advertising function where, for given amounts of advertising, only a small share of the market remains uninformed. We see that, without regard to price dispersion, cost asymmetries, or even search costs, if advertising is sufficiently effective, the low price firm always takes excessive advantage. Although more direct, this result coincides with our intuition for part (i).

We now turn to under-advertising as discussed in part (iii). In this case, advertising is prohibitively expensive for the firm (as given by $A \geq \bar{A}$) and profit margins are sufficiently close (i.e., $c \leq \underline{c}$). These conditions work together to dull the advertising incentives of the firm, so much that the firm under-advertises relative to a planner. The requirement that $\frac{1-Q(c)}{q(c)} < c$ is essentially a hazard rate condition that places an upper bound on the high cost firm's profit margin even for high price dispersion. This ensures that even if profit margins are identical, the social benefit of advertising due to decreased expenditures on costs of production exceeds the private benefit of advertising from attracting more high search cost consumers.

Having discussed the incentives of the firm, we now consider those of the planner. The key here is that changes in advertising have a uniform effect on

the market, while changes in price dispersion only affect consumers with search costs in a particular range. In certain cases, the uniform effect of a marginal change in advertising has a larger impact on the average search cost consumer, to whom the planner is concerned, than on high search cost consumers, to whom the advertising firm is concerned. More specifically, over- or under-advertising depends on the share of high search cost consumers in the market as this share determines the effect of advertising relative to search, where the planner would rather prohibit search if the indifferent consumer has a high search cost ($\hat{s} > \mu$) and would be more willing to allow search otherwise.

For instance, if advertising is expensive and costs of production are low, then margins differences are small and \hat{s} must be close to c . In this case, an increase in x would decrease \hat{s} so that a marginal proportion of relatively high search cost consumers no longer pay their search cost. This proportion of newly-inactive searchers, however, might now have to fund the cost of production c because, due to the uniform effect of advertising, there is no guarantee that the extra advertising would reach these same consumers. But to the planner, this tradeoff is relatively meaningless since \hat{s} and c are close. More generally, the social benefit of advertising due to decreased search cost expenditures is high since \hat{s} is relatively high. It follows, then, that the firm under-advertises relative to a planner.

This intuition also applies when firms over-advertise. For large margin differences, consumer search is only a small determinant of firm demand, and the indifferent consumer has a low search cost relative to c . In this case, *(i)* the uniform effect of a change in advertising has a potentially large effect on firm demand relative to search, and *(ii)* only a small portion of (low search cost) consumers actually pay the search cost to visit the low price firm. Similar to the example above, an increase in x would decrease \hat{s} , making a marginal proportion of low search cost consumers inactive searchers. But there is again no guarantee that the extra advertising would reach that portion of the market. In essence, an increase in x risks sending low search cost consumers away from the low price firm (only

a small social gain due to decreased search costs) and forcing them to fund the relatively high costs of production. In this case, the planner would rather use search to send more relatively low search cost buyers to the low price firm than rely on the uniform effects of advertising, in which case the firm over-advertises.

Note that, for symmetric search cost distributions, the indifferent consumer having a high search cost is equivalent to a market composed primarily of active searchers. For such distributions, we therefore conclude that firms over-advertise when inactive searchers compose the majority of the market and vice versa for active searchers. This does not hold for all distributions, however, as a highly skewed $q(\cdot)$ could imply a large proportion of consumers search while the indifferent consumer's search cost remains small.

Finally, since all functions are continuous, and since both under- or over-advertising can result, there must be some combination of distributions, functional form specifications, and cost parameters such that the interests of both the firm and the planner align. Although this is a knife-edge situation, it is interesting in that the two firms, acting purely in self-interest, could reach the socially optimal outcome.

4 Conclusion

The imperfect nature of price information in search models provides a natural framework within which to study price advertising. Previous studies, however, have not offered definitive welfare results under optimal consumer and firm behavior. This is a nontrivial issue as the planner and firm have potentially conflicting definitions of the value of advertising. In this paper, we put enough structure on the market to explicitly compare optimal and duopolistic advertising levels. We do so in an equilibrium search setting where we impose the duopolistic price level on the planner's problem. Our analysis explains well the relationship between the firm's and the planner's incentives to advertise.

We find that firms might under- or over-advertise relative to a planner and that the result depends on several factors—primarily the effectiveness of advertising and the costs of production. Specifically, we find that firms place significantly more weight on the informational role of advertising whenever profit margins are sufficiently different. This means that when production cost asymmetries are large so that the social and private values of advertising are high, it is the firm, not the planner, that takes advantage. Conversely, we find that firms under-advertise when advertising costs are high and margins are close and subsequently place too much weight on the role of search in attracting buyers.

In particular, we get both under- and over-advertising in a setting where advertising is purely informative and without focusing on many identical firms. We do so in the context of an equilibrium consumer search model where *(i)* advertising has an obvious role in forming and improving buyers' knowledge of prices and *(ii)* where advertising and search are imperfect substitutes for transmitting price information. Our approach shows that the welfare effects of advertising are not a strict byproduct of the type of advertising in question, the elasticity of demand, or the nature of competition among firms.

A Proof of Proposition 1

By assumption, all functions are continuous and strategy sets are compact intervals. Therefore, by the standard Nash-Debreu theorem, a restricted pure-strategy Nash equilibrium exists so long as profit functions are quasi-concave in own strategy variables. From (2.9),

$$\begin{aligned} \frac{\partial^2 \pi_H}{\partial p_H^2} &= \frac{1}{2} f(x)(1 - Q(\hat{s})) \Pi_{pp}^H - \frac{1}{2} f(x) d(p_H) [q(\hat{s}) \Pi_p^H + q'(\hat{s}) d(p_H) \Pi^H] \\ &\quad - \frac{1}{2} f(x) q(\hat{s}) [d(p_H) \Pi_p^H + d'(p_H) \Pi^H]. \end{aligned} \quad (\text{A.1})$$

By previous assumptions on the monopolists problem, the hazard condition, and on the advertising function, we know that (A.1) is negative, so π_H is strictly concave. By these same conditions,

$$\frac{\partial^2 \pi_L}{\partial p_L \partial x} = -\frac{1}{2} f'(x) [(1 - Q(\hat{s})) \Pi_p^L + q(\hat{s}) d(p_L) \Pi^L] > 0, \quad (\text{A.2})$$

$$\frac{\partial^2 \pi_L}{\partial x^2} = -\frac{1}{2} f''(x) (1 - Q(\hat{s})) d(p_L) p_L \leq 0, \text{ and} \quad (\text{A.3})$$

$$\begin{aligned} \frac{\partial^2 \pi_L}{\partial p_L^2} &= \left[1 - \frac{1}{2} f(x) (1 - Q(\hat{s})) \right] \Pi_{pp}^L \\ &\quad - \frac{1}{2} f(x) d(p_L) [q(\hat{s}) \Pi_p^L - q'(\hat{s}) d(p_L) \Pi^L] \\ &\quad - \frac{1}{2} q(\hat{s}) f(x) [d(p_L) \Pi_p^L + d'(p_L) \Pi^L] < 0. \end{aligned} \quad (\text{A.4})$$

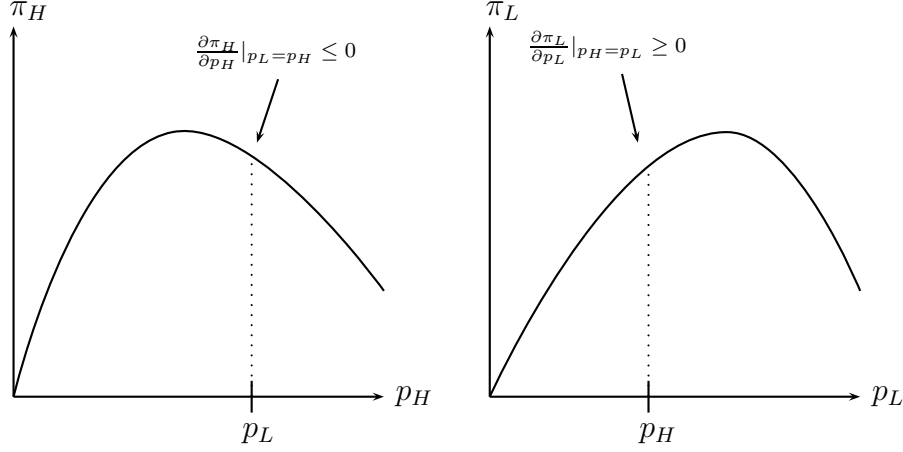
Therefore, the determinant of the bordered Hessian for the low cost firm must be positive, which then implies that π_L is quasi-concave. ■

B Proof of Proposition 2

FIRST PROVE (I)

First note that, from $\Pi = d(p)(p - c)$, we know that for any common price $p_H = p_L = p$, $\Pi_p^H|_{p_H=p} = d'(p)(p - c) + d(p) > d'(p)p + d(p) = \Pi_p^L|_{p_L=p}$ for all $c > 0$. Now suppose there exists a restricted Nash equilibrium that is not a Nash equilibrium. In such a case, at least one player is not making a best response. Figure 2 represents a graphical example of such a situation, where at least one firm would like to deviate from the restricted pricing strategy for a given advertising intensity x .

Figure 2: PROFIT FUNCTIONS



If the high cost firm is not making a best response, then $\frac{\partial \pi_H}{\partial p_H} |_{p_H=p_L} < 0$, while if the low cost firm is not making a best response, $\frac{\partial \pi_L}{\partial p_L} |_{p_L=p_H} > 0$. In either case, it must be that $\frac{\partial \pi_H}{\partial p_H} |_{p_H=p_L} \leq 0$ and $\frac{\partial \pi_L}{\partial p_L} |_{p_L=p_H} \geq 0$, where it follows that the restricted equilibrium must be at $p_H = p_L$, which implies that $x^* = 0$, $f(x^*) = 1$, and $Q(\hat{s}) = Q(0) = 0$. The resulting first order conditions are as follows (where $p_L = p_H = p$):

$$\begin{aligned} \frac{\partial \pi_H}{\partial p_H} &= \frac{1}{2} \Pi_p^H |_{p_H=p} - \frac{1}{2} q(0) d(p)^2 (p - c), \text{ and} \\ \frac{\partial \pi_L}{\partial p_L} &= \frac{1}{2} \Pi_p^L |_{p_L=p} - \frac{1}{2} q(0) d(p)^2 p. \end{aligned}$$

Since $\Pi_p^H > \Pi_p^L$ from before, we see that

$$\frac{\partial \pi_H}{\partial p_H} > \frac{\partial \pi_L}{\partial p_L}$$

must hold for all $c > 0$. Without loss of generality, assume $\frac{\partial \pi_H}{\partial p_H} \leq 0$. Then it must be that

$$\frac{\partial \pi_L}{\partial p_L} < \frac{\partial \pi_H}{\partial p_H} \leq 0.$$

This cannot be a restricted equilibrium as $\frac{\partial \pi_L}{\partial p_L} < 0$, and the low cost firm wants to decrease price.

NOW PROVE (II)

From (I), we know that $p_H^* = p_L^*$ cannot hold in equilibrium, so we need only consider the case

where $p_H^* < p_L^*$. Denote each firm's monopoly price by $p_M(c) = \max(p - c)d(p)$, so that $p_M(0)$ is the monopoly price of the low cost firm. In any Nash equilibrium, $p_L^* < p_M(0) < p_{max}$. Then for $p_{max} > c > p_M(0)$, the high cost firm never prices below p_L . Accordingly, there exists some $\bar{c} > 0$, $\bar{c} < p_{max}$, such that, for all $c > \bar{c}$, every Nash equilibrium involves the low cost firm pricing below the high cost firm. ■

C Proof of Proposition 3

Assume that firms adjust their strategies according to

$$\frac{da_i}{dt} = k_i \frac{\partial \pi_i(a_1, a_2, a_3)}{\partial a_i} \quad (\text{C.1})$$

in a neighborhood of the equilibrium. In the usual way, we take a first-order Taylor approximation and, ignoring the constants k_i , we find

$$\begin{bmatrix} \frac{dp_L}{dt} \\ \frac{dp_H}{dt} \\ \frac{dx}{dt} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 \pi_L(p_L^*, p_H^*, x^*)}{\partial p_L^2} & \frac{\partial^2 \pi_L(p_L^*, p_H^*, x^*)}{\partial p_L \partial p_H} & \frac{\partial^2 \pi_L(p_L^*, p_H^*, x^*)}{\partial p_L \partial x} \\ \frac{\partial^2 \pi_H(p_L^*, p_H^*, x^*)}{\partial p_H \partial p_L} & \frac{\partial^2 \pi_H(p_L^*, p_H^*, x^*)}{\partial p_H^2} & \frac{\partial^2 \pi_H(p_L^*, p_H^*, x^*)}{\partial p_H \partial x} \\ \frac{\partial^2 \pi_L(p_L^*, p_H^*, x^*)}{\partial x \partial p_L} & \frac{\partial^2 \pi_L(p_L^*, p_H^*, x^*)}{\partial x \partial p_H} & \frac{\partial^2 \pi_L(p_L^*, p_H^*, x^*)}{\partial x^2} \end{bmatrix} \begin{bmatrix} p_L - p_L^* \\ p_H - p_H^* \\ x - x^* \end{bmatrix}.$$

We need to show that the real parts of all eigenvalues are negative, which will ensure that our system is stable. A sufficient condition, therefore, is that our Hessian matrix has a dominant diagonal. By definition, any $n \times n$ matrix A has a dominant diagonal if there exists some $d_i > 0$, for $i = 1, 2, \dots, n$, such that $d_i |\pi_{ii}| > \sum_{j \neq i} d_j |\pi_{ij}|$.

For convenience, denote the following

$$\begin{aligned} \lambda_{11} &= (\text{A.4}), \\ \lambda_{12} &= \frac{1}{2} f(x) d(p_H) [q(\hat{s}) \Pi_p^L - q'(\hat{s}) d(p_L) \Pi^L], \\ \lambda_{13} &= (\text{A.2}), \\ \lambda_{21} &= \frac{1}{2} f(x) d(p_L) [q(\hat{s}) \Pi_p^H + q'(\hat{s}) d(p_H) \Pi^H], \\ \lambda_{22} &= (\text{A.1}), \\ \lambda_{23} &= \frac{1}{2} f'(x) [(1 - Q(\hat{s})) \Pi_p^H - q(\hat{s}) d(p_H) \Pi^H], \\ \lambda_{31} &= (\text{A.2}), \\ \lambda_{32} &= \frac{1}{2} f'(x) q(\hat{s}) d(p_H) \Pi^H, \text{ and} \\ \lambda_{33} &= (\text{A.3}). \end{aligned}$$

Denote the matrix with the above elements by Λ . Sufficient conditions under elastic demand are complicated and omitted for space. It can be shown, however, that such conditions are maximized under inelastic demand. Accordingly, we consider unit inelastic demand to show that Λ has a dominant diagonal. After imposing the first order conditions for price and setting $d_1 = d_2 = d_3 = 1$, the following three conditions are sufficient for a dominant diagonal and thus

stability:

$$\begin{aligned}
& -\frac{1}{2}f(x)q(\hat{s}) - \frac{f'(x)}{f(x)} < 0, \\
& -\frac{1}{2}f(x)q(\hat{s}) < 0, \text{ and} \\
& -\frac{1}{2}f''(x)(1 - Q(\hat{s}))p_L - \frac{f'(x)}{f(x)} - \frac{1}{2}f'(x)q(\hat{s})p_L < 0.
\end{aligned}$$

These hold so long as

$$\frac{f'(x)}{f(x)} > -\frac{1}{2}f(x)q(\hat{s}), \text{ and} \quad (\text{C.2})$$

$$\frac{f''(x)}{f'(x)} < -\frac{1}{1 - Q(\hat{s})} \left[\frac{2}{p_L f(x)} + q(\hat{s}) \right]. \quad (\text{C.3})$$

Using equilibrium conditions

$$\begin{aligned}
p_L &= \frac{2}{f(x)q(\hat{s})} - \frac{1 - Q(\hat{s})}{q(\hat{s})}, \text{ and} \\
p_H - c &= \frac{1 - Q(\hat{s})}{q(\hat{s})},
\end{aligned}$$

we see that $\hat{s} = \frac{2}{q(\hat{s})} \left[1 - Q(\hat{s}) - \frac{1}{f(x)} \right] + c$, which implies that \hat{s} is bounded above by c . We also see that $f(x)$ is bounded below by $\frac{2}{(2p_{max} - c)q_{max}}$ and that $q(\hat{s})$ is bounded below by $\frac{1 - Q(\hat{s})}{p_{max} - c}$. Therefore, assuming $Q(c) < 1$ provides an upper bound of $Q(\hat{s})$ and a lower bound on $1 - Q(\hat{s})$, and we can rewrite the above conditions as

$$\frac{f'(x)}{f(x)} > -\frac{(1 - Q(c))}{(2p_{max} - c)(p_{max} - c)q_{max}}, \text{ and} \quad (\text{C.4})$$

$$\frac{f''(x)}{f'(x)} < -\frac{3q_{max}}{1 - Q(c)}. \quad (\text{C.5})$$

Therefore, under conditions (C.4) and (C.5), Λ has a dominant diagonal, and the adjustment process defined by (C.1) is locally stable. Note that the expression for (C.4) given in the text is a slightly stronger sufficient condition. ■

D Proof of Proposition 4

Totally differentiating the system of first order conditions formed by (2.6), (2.9), and (2.7) with respect to p_L , p_H , x , A , and c provides the system of equations with which to derive comparative statics. Recalling Λ above, the differentiated system can then be written as follows:

$$\Lambda \begin{bmatrix} dp_L \\ dp_H \\ dx \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2}f(x)q(\hat{s})d(p_H)^2dc \\ dA \end{bmatrix}.$$

Just as in Appendix C, we impose the first order conditions for price, which greatly simplifies λ_{23} and λ_{13} . We also again consider the inelastic demand case for brevity, where it can be shown

that the determinant is maximized under this setting. This yields

$$\begin{aligned} |\Lambda| &= -\frac{1}{8}f''(x)f(x)^2(1-Q(\hat{s}))p_L(2q(\hat{s})-q'(\hat{s})p_L)(2q(\hat{s})+q'(\hat{s})(p_H-c)) \\ &\quad +\frac{1}{8}f''(x)f(x)^2(1-Q(\hat{s}))p_L(q(\hat{s})+q'(\hat{s})(p_H-c))(q(\hat{s})-q'(\hat{s})p_L) \\ &\quad -\frac{f'(x)^2}{2f(x)}[f(x)q(\hat{s})p_L(q(\hat{s})+q'(\hat{s})(p_H-c))-(2q(\hat{s})+q'(\hat{s})(p_H-c))]. \end{aligned}$$

Imposing stability conditions (C.2) and (C.3), it follows that $|\Lambda| < 0$, and applying Cramer's rule, we find

$$\begin{aligned} \frac{dp_L}{dc} &= \frac{\frac{1}{2}f(x)q(\hat{s})\left[-\frac{1}{4}f(x)f''(x)(1-Q(\hat{s}))p_L[q(\hat{s})-q'(\hat{s})p_L]+\frac{f'(x)^2}{2f(x)}q(\hat{s})p_L\right]}{|\Lambda|} \geq 0, \\ \frac{dp_L}{dA} &= -\frac{\frac{1}{2}f'(x)[2q(\hat{s})+q'(\hat{s})(p_H-c)]}{|\Lambda|} \leq 0, \\ \frac{dp_H}{dc} &= -\frac{\frac{1}{2}f(x)q(\hat{s})\left[\frac{1}{4}f''(x)f(x)(1-Q(\hat{s}))p_L[2q(\hat{s})-q'(\hat{s})p_L]-\frac{f'(x)^2}{f(x)^2}\right]}{|\Lambda|} \geq 0, \\ \frac{dp_H}{dA} &= -\frac{\frac{1}{2}f'(x)[q(\hat{s})+q'(\hat{s})(p_H-c)]}{|\Lambda|} \leq 0, \\ \frac{dx}{dc} &= \frac{\frac{1}{4}f'(x)f(x)q(\hat{s})\left[-\frac{1}{2}f(x)q(\hat{s})p_L[2q(\hat{s})-q'(\hat{s})p_L]+(q(\hat{s})-q'(\hat{s})p_L)\right]}{|\Lambda|} \leq 0, \\ \frac{dx}{dA} &= \frac{\frac{1}{4}f(x)^2[(2q(\hat{s})-q'(\hat{s})p_L)(2q(\hat{s})+q'(\hat{s})(p_H-c))]}{|\Lambda|} \\ &\quad -\frac{\frac{1}{4}f(x)^2[(q(\hat{s})-q'(\hat{s})p_L)(q(\hat{s})+q'(\hat{s})(p_H-c))]}{|\Lambda|} \leq 0. \end{aligned}$$

With respect to advertising, we treat x as exogenous and derive $\frac{d\hat{s}}{dx}$ in the usual way. Again looking at the inelastic demand case, totally differentiating (2.6) and (2.9) with respect to p_H , p_L , x , and c yields the following system

$$\begin{bmatrix} -q'(\hat{s})(p_H-c)-2q(\hat{s}) & q'(\hat{s})(p_H-c)+q(\hat{s}) \\ \frac{1}{2}f(x)(q(\hat{s})-q'(\hat{s})p_L) & -\frac{1}{2}f(x)(2q(\hat{s})-q'(\hat{s})p_L) \end{bmatrix} \begin{bmatrix} dp_H \\ dp_L \end{bmatrix} = \begin{bmatrix} -q(\hat{s})dc \\ \frac{1}{2}f'(x)(1-Q(\hat{s})+q(\hat{s})p_L)dx \end{bmatrix}.$$

For simplicity, define the following matrices:

$$\begin{aligned} \Omega &= \begin{bmatrix} -q'(\hat{s})(p_H-c)-2q(\hat{s}) & q'(\hat{s})(p_H-c)+q(\hat{s}) \\ \frac{1}{2}f(x)(q(\hat{s})-q'(\hat{s})p_L) & -\frac{1}{2}f(x)(2q(\hat{s})-q'(\hat{s})p_L) \end{bmatrix}, \\ \Omega_{p_H} &= \begin{bmatrix} -q(\hat{s})dc & q'(\hat{s})(p_H-c)+q(\hat{s}) \\ \frac{1}{2}f'(x)(1-Q(\hat{s})+q(\hat{s})p_L)dx & -\frac{1}{2}f(x)(2q(\hat{s})-q'(\hat{s})p_L) \end{bmatrix}, \text{ and} \\ \Omega_{p_L} &= \begin{bmatrix} -q'(\hat{s})(p_H-c)-2q(\hat{s}) & -q(\hat{s})dc \\ \frac{1}{2}f(x)(q(\hat{s})-q'(\hat{s})p_L) & \frac{1}{2}f'(x)(1-Q(\hat{s})+q(\hat{s})p_L)dx \end{bmatrix}. \end{aligned}$$

From conditions (2.10) and (A.1), we know that

$$|\Omega| = \frac{1}{2}f(x)q(\hat{s})[q'(\hat{s})(p_H-p_L-c)+3q(\hat{s})] > 0.$$

Looking only at p_H , we see that

$$dp_H = \frac{[2q(\hat{s}) - q'(\hat{s})p_L] dc}{q'(\hat{s})(p_H - p_L - c) + 3q(\hat{s})} - \frac{f'(x) [1 - Q(\hat{s}) + q(\hat{s})p_L] [q'(\hat{s})(p_H - c) + q(\hat{s})] dx}{f(x)q(\hat{s}) [q'(\hat{s})(p_H - p_L - c) + 3q(\hat{s})]}.$$

Setting dc to zero, we find

$$\frac{dp_H}{dx} = \frac{-f'(x) [1 - Q(\hat{s}) + q(\hat{s})p_L] [q'(\hat{s})(p_H - c) + q(\hat{s})]}{f(x)q(\hat{s}) [q'(\hat{s})(p_H - p_L - c) + 3q(\hat{s})]}.$$

The same process for p_L yields

$$\frac{dp_L}{dx} = \frac{-f'(x) [1 - Q(\hat{s}) + q(\hat{s})p_L] [q'(\hat{s})(p_H - c) + 2q(\hat{s})]}{f(x)q(\hat{s}) [q'(\hat{s})(p_H - p_L - c) + 3q(\hat{s})]}.$$

Therefore, we know

$$\frac{d\hat{s}}{dx} = \frac{dp_H}{dx} - \frac{dp_L}{dx} = \frac{f'(x) [1 - Q(\hat{s}) + q(\hat{s})p_L]}{f(x) [q'(\hat{s})(p_H - p_L - c) + 3q(\hat{s})]} < 0. \blacksquare$$

E Proof of Proposition 5

Since π_L and π_H are concave in p_L and p_H respectively, the first order conditions for p_L and p_H are necessary and sufficient for a constrained optimum of the planner's problem. We can then solve the constraints implicitly for $\hat{s}(x)$ and plug this into the objective function. To ensure a unique optimum, we need only show that the resulting function is strictly concave.

Differentiating the welfare function and rearranging terms yields

$$\begin{aligned} \frac{d^2W}{dx^2} &= -\frac{1}{2}f''(x) \left[(1 - Q(\hat{s}))c + \int_0^{\hat{s}} sq(s)ds \right] \\ &\quad - f'(x)q(\hat{s})(\hat{s} - c) \frac{d\hat{s}}{dx} - \frac{1}{2}f(x)q(\hat{s}) \left(\frac{d\hat{s}}{dx} \right)^2 \\ &\quad - \frac{1}{2}f(x)(\hat{s} - c) \left[q'(\hat{s}) \left(\frac{d\hat{s}}{dx} \right)^2 + q(\hat{s}) \frac{d^2\hat{s}}{dx^2} \right]. \end{aligned} \tag{E.1}$$

After substituting $\frac{d\hat{s}}{dx}$ and $\frac{d^2\hat{s}}{dx^2}$ and some tedious algebra, we see that (E.1) is always negative provided $\frac{q''(\hat{s})}{q(\hat{s})} \leq \frac{-f''(x)}{f'(x)(2p_{max}-c)c} - \frac{2}{(p_{max}-c)c}$, and there exists a unique socially optimal advertising level subject to the equilibrium duopoly price level. \blacksquare

F Proof of Proposition 6

FIRST PROVE (I)

For convenience, denote $\phi = \int_0^{\hat{s}} sq(s)ds + (1 - Q(\hat{s}))(c - p_L)$, then substituting the firm's first

order condition for advertising, (2.7), yields

$$\begin{aligned}
\frac{dW}{dx}|_{x^d} &= -\frac{1}{2}f'(x)\phi - \frac{1}{2}f(x)q(\hat{s})(\hat{s} - c)\frac{d\hat{s}}{dx} \\
&= -\frac{1}{2}f'(x)\phi - \frac{1}{2}f'(x)q(\hat{s})(\hat{s} - c)\frac{1 - Q(\hat{s}) + q(\hat{s})p_L}{q'(\hat{s})(p_H - p_L - c) + 3q(\hat{s})} \\
&= -\frac{1}{2}f'(x)\phi - q(\hat{s})\frac{f'(x)(\hat{s} - c)}{f(x)[q'(\hat{s})(p_H - p_L - c) + 3q(\hat{s})]}
\end{aligned}$$

where the third equality comes from substituting (2.6). From this equation, we know the sign of $\frac{dW}{dx}|_{x^d}$ depends on ϕ . To see this, note that equilibrium first order conditions, (2.6) and (2.9), require

$$\hat{s} - c = p_H - p_L - c = \frac{2}{q(\hat{s})} [1 - Q(\hat{s}) - f^{-1}(x)]$$

which is nonpositive as $f(x) \in [0, 1]$ and $Q(\hat{s}) \geq 0$. Also, from (2.10) we know $q'(\hat{s})(p_H - p_L - c) + 3q(\hat{s}) > 0$. So the following results hold:

$$\frac{dW}{dx}|_{x^d} > 0 \text{ iff } \phi > \frac{-2q(\hat{s})(\hat{s} - c)}{f(x)[q'(\hat{s})(\hat{s} - c) + 3q(\hat{s})];} \quad (\text{F.1})$$

$$\frac{dW}{dx}|_{x^d} \leq 0 \text{ iff } \phi \leq \frac{-2q(\hat{s})(\hat{s} - c)}{f(x)[q'(\hat{s})(\hat{s} - c) + 3q(\hat{s})]}. \quad (\text{F.2})$$

We proceed by examining the comparative statics of \hat{s} to changes in c and A as well the upper bound of ϕ to determine when equation (F.2) holds.

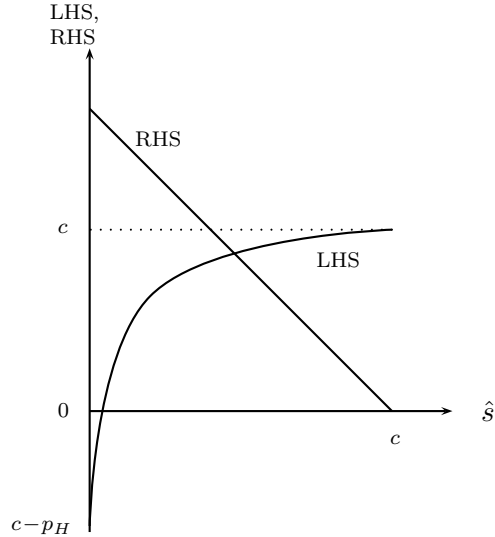
First consider the upper bound of ϕ . From equations (2.6) and (2.9), we see that $c - p_L = p_H - \frac{2}{q(\hat{s})f(x)}$, which is bounded above by $p_{max} - \frac{2}{q_{max}}$. This implies that ϕ is bounded above by $\mu + p_{max} - \frac{2}{q_{max}}$, where μ is the mean of s (note that $\mu \geq \int_0^{\hat{s}} sq(s)ds$). Therefore, for $p_{max} < \frac{2}{q_{max}} - \mu$, it follows that $\frac{dW}{dx}|_{x^d} \leq 0$.

Now consider comparative statics. First, we rewrite the upper bound of ϕ by noting that $\int_0^{\hat{s}} sq(s)ds \leq \hat{s}Q(\hat{s})$, which implies that $\phi \leq c + Q(\hat{s})(\hat{s} - c)$. Therefore, over-advertising results for

$$c + Q(\hat{s})(\hat{s} - c) \leq -\frac{2(\hat{s} - c)}{f(x)\left[\frac{q'(\hat{s})}{q(\hat{s})}(\hat{s} - c) + 3\right]}. \quad (\text{F.3})$$

We use both equations (F.2) and (F.3) in the following. Note that any equilibrium requires $\hat{s} \leq 1$. Otherwise, the low price firm could increase price to $p_H - 1$ and still get the entire market. Since $\hat{s} - c$ is always negative in equilibrium, this implies that $\min\{1, c\} \geq \hat{s} \geq 0$. Now consider the left and right hand sides of equation (F.2) as \hat{s} goes to its maximum, where we see that the left hand side is bounded above by c for $c < 1$ and bounded above by μ for $c > 1$, and the right hand side is equal to 0 for $c < 1$ and is positive for $c > 1$. For the lower bound of \hat{s} ($\hat{s} = 0$), we see that $\phi = c - p_H < 0$ as $\hat{s} = 0 \Rightarrow p_L = p_H$. Since the right hand side is positive for $\hat{s} = 0$, it follows that equation (F.2) is always satisfied for $\hat{s} = 0$ and unsatisfied for $\hat{s} = c < 1$. Figure 3 describes these bounds graphically, where over-advertising is depicted by the range in which RHS is above LHS. We only consider graphically the case where $c < 1$, but a similar result holds for $c > 1$ as ϕ remains bounded above by μ .

Figure 3: OVER-ADVERTISING



While the left and right hand sides are increasing in both \hat{s} and c and the shape of the functions themselves also changes, the bounds remain fixed as in figure 3. Since all functions are continuous, it follows that there exists some \hat{s} such that over-advertising results for all $\hat{s} \leq \hat{s}$. From Proposition 4, we see that \hat{s} is increasing in A . Therefore, we know there exists some \underline{A} such that, for all $A \leq \underline{A}$, the duopolistic advertising level exceeds the social optimum. Note that, although \hat{s} is also increasing in c , we cannot make a similar statement for small c because the right hand side of equation (F.2) shifts down at a faster rate than the left hand side as c decreases (under certain assumptions consistent with those needed for existence of a social optimum), making this inequality much less likely for small c .

Finally, recall that $\hat{s} \rightarrow 1$ as c increases, in which case $\phi = \int_0^{\hat{s}} sq(s)ds + (1-Q(\hat{s}))(c-p_L) \rightarrow \mu$ since $Q(1) = 1$. In this case, the right hand side of equation (F.2) is increasing in c while the left hand side does not change, so there exists some \bar{c} such that over-advertising results for all $c \geq \bar{c}$.⁸ This proves part (i).

NOW PROVE (II)

Since the upper bound for ϕ is independent of x , and since the lower bound for

$$\frac{-2q(\hat{s})(\hat{s} - c)}{f(x)[q'(\hat{s})(\hat{s} - c) + 3q(\hat{s})]}$$

is increasing in x , it follows that over-advertising results for $f(x) \in [0, 1]$ sufficiently small. We formally characterize this by considering $f(t, x)$, $t > 0$, in which case $f_t < 0$ and t sufficiently large implies that $f(x)$ is small even for small levels of advertising. Finally, we restrict problems along the boundary by assuming that, as $x \rightarrow 0$, f_x becomes large. This proves part (ii).

NOW PROVE (III)

Recall that over-advertising occurs iff $\phi > \frac{-2q(\hat{s})(\hat{s} - c)}{f(x)[q'(\hat{s})(\hat{s} - c) + 3q(\hat{s})]}$. Following a similar process as

⁸We consider the more extreme argument of $\hat{s} \rightarrow 1$ so as to avoid the extra assumptions required to ensure that the right hand side of equation (F.2) shifts up at a faster rate than the left hand side as c increases, although it can be show that under assumptions consistent with existence of a social optimum, such a result holds.

with part (i), $\hat{s} \rightarrow c$ as A increases, which implies that $p_H - c = p_L$. This also implies that $(1 - Q(\hat{s}))(c - p_L) \rightarrow (1 - Q(c))(c - (p_H - c))$ as $\hat{s} \rightarrow c$.

Now consider $(1 - Q(c))(2c - p_H)$ as a lower bound for ϕ at $\hat{s} = c$. To ensure that this is positive, we need $\frac{1 - Q(c)}{q(c)} < c$, in which case ϕ is positive at this lower bound and the right hand side goes to zero. Therefore, provided c is relatively small, equation (F.1) will hold for some A sufficiently large since \hat{s} is increasing in A . So, there exists some cost pair (\bar{A}, \underline{c}) such that for all $A \geq \bar{A}$ and $c \leq \underline{c}$ the duopolistic advertising level is below that of a planner. This proves part (iii). ■

G Proof of Corollary

Since the welfare function is continuous in all variables and strictly concave in x , and since the proof for Proposition 6 shows that it is possible for both $\frac{dW}{dx}|_{x^d} > 0$ and $\frac{dW}{dx}|_{x^d} \leq 0$ depending on functional and parameter specifics, it follows that there is some intermediate value of x such that $\frac{dW}{dx}|_{x^d} = 0$. By strict concavity, this advertising level must be such that the duopolistic and socially optimal advertising levels are the same. ■

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