Abstract This paper considers a small open economy where government pursues debt targeting policy. In this setting, I find the optimal debt targeting rule crucially hinges on the assumption of bond market structure, i.e. whether household can purchase state-contingent bond. In addition, the country’s foreign asset holding and government’s debt level may have substantial impact on the optimal debt targeting rule. Furthermore, I derive analytical expressions for the fiscal sustainability condition and find it depends on the fiscal adjustment speed and the tax revenue elasticity.

1 Introduction

Most OECD countries have adopted fiscal rules to safeguard budget sustainability. Well known is the rule of Maastricht Treaty in the EU countries (1992, extended in 1997). The government net borrowing is limited to 3 percent of GDP, and the gross debt should not exceed the ceiling of 60 percent of GDP. Appendix A lists the debt rules in some other selected OECD countries. Taxes have to be adjusted when government is unable or unwilling to meet the requirement through cutting government spending.

However, in theory little has been said about the welfare consequence of pursuing such fiscal dislines. Large amount of interest payment caused by high debt issuance leaves little room for government to balance budget. On the other hand, it is well known that in a closed economy tax should be smoothed across time or over state, and the debt should be the shock absorber (Lucas and Stokey (1983), Barro (1979), Aiyagari et al (2002)). When government constrains the debt issuance, it loses the intrustment of tax to smooth consumption. Facing this tradeoff, it is of interest to investigate the optimal debt rule.

In addition, in the literature of optimal fiscal policy within closed economy, it is typical to assume that private agent can trade state-contingent bond among them. This assumption has been extended into small open economy(Schmitt-Grohe and Uribe (2003a), Angyridis (2006)), which implies that state-contingent bond plays a role of perfect insurance and household is able to insure against any shock through international market. However, a more realistic scenario is that people can not be fully insured. By comparing a complete asset market (household can hold state-contingent bond) and an incomplete asset market (household can only purchase risk-free bond), this paper will analyze to what extent the bond market structure changes the optimal debt rule.

*This draft is preliminary. I am grateful to Eric Leeper for inspiring my interest on this topic. I thank Eric Leeper, Brian Peterson and Edward Buffie for many suggestions. Also, I thank Giorgio Primiceri for helpful conversation. All errors are mine. Department of Economics, Indiana University, hbi@indiana.edu.
To address those questions, I use a real small open economy model. As a first step on this topic, I assume the government adopts a linear debt targeting rule, instead of nonlinear debt limit rule as most OECD countries are pursuing. Household may or may not be able to hold state-contingent bond, depending on the model specification.

I find that under some pausable calibration the optimal debt targeting rule could be completely opposite between the incomplete and complete asset market. With state-contingent bond, household can completely smooth their consumption through international market, and their welfare only depends on leisure. That is not the case in the world with only risk-free bond. People face the tradeoff between smoothing consumption and enjoying more leisure. This can lead to fundamentally different results.

Furthermore, when household can only hold risk-free bond, the country’s foreign asset holding and government’s fiscal position have substantial impact on the optimal rule. If the government has issued large amount of debt, or if the country is a net borrower, high country risk premium makes it very costly for household to further borrow from foreigners. This constrains people from smoothing consumption through international market. More rapid fiscal adjustment may make people better off. On the other hand, if the government is in a sound fiscal position, or if the country is a net lender, household can lean upon foreigners in hard times. Less rapid adjustment helps to smooth the convergency path and may improve welfare.

In addition, the fiscal sustainability conditions are derived analytically in the economy with complete asset market. It shows that too sluggish adjustment leads to explosive debt path, while too rapid adjustment causes overshooting. The quantitative range of adjustment speed depends on tax revenue elasticity.

The paper is organized as following. Section 2 studies fiscal sustainability conditions. Section 3 discusses the optimal debt targeting rule. Section 4 concludes.

2 Fiscal sustainability

In order to derive the fiscal sustainability conditions analytically, I use a model where household have access to complete asset market, in the sense that a representative household can invest in a set of state-contingent one-period real securities that span all the states of nature. In a small open economy, the access to complete international asset market implies constant marginal utility of consumption. This property can largely simplify the dynamics.

2.1 Complete asset market without capital

First consider a model where household can hold state-contingent bond, but not capital.

2.1.1 Household

The state variable $s^t = (s_0, s_1, ..., s_t)$ is a random event that is an element of a finite set $S$. The state $s^t$ is determined by shocks to government spending. The probability at period 0 of any particular history $s^t$ being realized is denoted by $\pi(s^t)$. The initial state is given.
At period $t$, the household can trade any one-period forward Arrow securities of market value $p(s_{t+1}|s^t) b(s_{t+1}|s^t)$ for wage normalized to 1. They also allocate consumption and labor according to

$$\max \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) u(c(s^t), L(s^t))$$ \hspace{1cm} (1)

subject to

$$c(s^t) + \sum_{s_{t+1}} p(s_{t+1}|s^t) b(s_{t+1}|s^t) = L(s^t) \left(1 - \tau(s^t)\right) + b(s^t)$$ \hspace{1cm} (2)

It can be shown that the optimization is equivalent to the following question.

$$\max E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, L_t)$$ \hspace{1cm} (3)

subject to

$$c_t + E_t q_{t+1} b_{t+1} = L_t (1 - \tau_t) + b_t$$ \hspace{1cm} (4)

$$q_{t+1} = p(s_{t+1}|s^t) / \pi(s_{t+1}|s^t)$$ \hspace{1cm} (5)

where $q_{t+1}$ denotes the period-$t$ price of an asset that pays one unit of good in a particular state of period $t + 1$ divided by the probability of occurrence of that state given information available in period $t$. Household first-order conditions are straightforward,

$$u_c(t) q_{t+1} = \beta u_c(t+1)$$ \hspace{1cm} (6)

$$-\frac{u_L(t)}{u_c(t)} = (1 - \tau_t^L) w_t$$ \hspace{1cm} (7)

Also, the allocation satisfies the transversality condition,

$$E_t \lim_{i \to \infty} Q_{t+i} b_{t+i} = 0$$ \hspace{1cm} (8)

where $Q_{t+i} = q_{t+1} q_{t+2} \ldots q_{t+i}$.

### 2.1.2 Government

Government collects uniform tax on labor income. They also issue risk-free debt to finance their unproductive spending,

$$d_t + \tau_t L_t = R_{t-1} d_{t-1} + g_t$$ \hspace{1cm} (9)

where $1/R_t = E_t q_{t+1}$. Also, the debt-targeting rule is

$$\ln \frac{\tau^L_t}{\tau} = \gamma \ln \frac{S_{t-1}}{S}$$ \hspace{1cm} (10)

where the coefficient $\gamma$ is the fiscal adjustment parameter. Larger $\gamma$ implies more aggressive tax increase responding to upsurge government spending. $S$ is the debt-GDP ratio, defined as

$$S_{t-1} = \frac{d_{t-1}}{L_{t-1}}$$ \hspace{1cm} (11)
Last, the government is subject to a no-Ponzi scheme of the form,

$$E_t \lim_{i \to \infty} Q_{t+i} d_{t+i} = 0$$  \tag{12}$$

In the open economy, no-Ponzi-game constraint on households no longer guarantees that the government is not running a Ponzi scheme against the rest of the world. I assume that a prerequisite for the government to access to international market is the satisfaction of a borrowing limit like Equation \[12\].

### 2.1.3 The rest of the world

In the rest of the world, agents have access to the same array of financial assets as in the domestic economy (Schmitt-Grohe and Uribe (2003b)). Thus one can obtain similar first-order condition of the foreign household as domestic household. Let starred letter denotes foreign variables, it is

$$u^*_c(t) q_{t+1} = \beta u^*_c(t + 1)$$  \tag{13}$$

Since the domestic and foreign households share the same subjective discount factor, the domestic and foreign first-order conditions yield

$$\frac{u_c(t)}{u_{c+1}(t)} = \frac{u^*_c(t)}{u^*_{c+1}(t)}$$  \tag{14}$$

It implies that the domestic marginal utility of consumption is proportional to its foreign counterpart. That is,

$$u_c(t) = \vartheta u^*_c(t)$$  \tag{15}$$

where $\vartheta$ is a constant parameter determining differences in wealth across countries. The domestic economy is assumed to be small, thus $u^*_c(t)$ is exogenous. To investigate the effects of domestic government spending shock, I assume $u^*_c(t)$ is constant at this moment. Therefore the interest rate is constant as well.

$$u_c(t) = \vartheta u^*_c \equiv \vartheta^*$$  \tag{16}$$

$$q_{t+1} = \beta$$  \tag{17}$$

### 2.1.4 Fiscal sustainability conditions

Due to the existence of state-contingent bond in the international market, households can insure against any shock and maintain the constant marginal utility level, that is $u_c = \vartheta^*$. The dynamic system can be simplified as

$$- \frac{u_L(t)}{\vartheta^*} = (1 - \tau_t)$$  \tag{18}$$

$$d_t + \tau_t L_t = \frac{d_{t-1}}{\beta} + g_t$$

$$\ln \frac{\tau_t}{\tau} = \gamma^t \ln \frac{S_{t-1}}{S}$$  \tag{19}$$
With explicit utility functional form, this model can be solved analytically. At each period \( t \), income tax rate \( \tau_t \) is predetermined due the debt-targeting rule, and consequently labor choice \( L_t \) is also predetermined. Thus the system can nailed down to one single equation including one endogenous variable \( S_t \) and the exogenous shock \( g_t \).

More specifically, assume the utility function is

\[
U(c, L) = \ln c - \frac{\chi L^{1+\sigma}}{1+\sigma} \tag{20}
\]

The system can be substituted into a single equation

\[
S_t + \tau_t (S_{t-1} - 1) = \frac{1}{\beta} \frac{L(S_{t-2})}{L(S_{t-1})} + \frac{g_t}{L(S_{t-1})} \tag{21}
\]

where \( \tau(S_{t-1}) \) and \( L(S_{t-1}) \) shows that tax rate and labor supply can be represented as debt-GDP ratio in previous period. To discuss the fiscal sustainability, Equation (21) can be log-linearized around the steady states. Appendix B includes the details.

It can be shown that there exist a unique and stable equilibrium if the fiscal adjustment parameter \( \gamma \) satisfies the both following conditions

\[
\left( \frac{1}{\beta} - 1 \right) \left( 1 - \epsilon^R \right) \frac{T d}{\epsilon^R} \gamma > \frac{1}{\beta} - 1 \tag{22}
\]

\[
\frac{1}{\beta} > \left( T d \epsilon^R + \epsilon^R - 1 \right) \gamma > \frac{1}{\beta} - 2 \tag{23}
\]

where \( T = \tau L \) is the tax revenue at steady state, \( \epsilon^R = \hat{T}/\hat{\tau} \) is the elasticity of tax revenue with respect to tax rate. Intuitively, too small \( \gamma \) implies too slow fiscal adjustment and leads to an explosive path of government debt. Too large \( \gamma \) leads to a regime where the path oscillates around the steady state and can’t converge.

In another word, the dynamic path starts to fluctuate if

\[
\left( T d \epsilon^R + \epsilon^R - 1 \right) \gamma > \frac{1}{\beta} \tag{24}
\]

The larger the elasticity \( \epsilon^R \), the easier to reach the regime with fluctuation. The result is intuitive. Larger \( \gamma \) implies more rapid adjustment to fiscal imbalancement. With elastic tax revenue, aggressive adjustment may cause tax revenue to jump up to such an extent that leads to government budget surplus. Responding to the surplus, government would cut the tax again, if they take the debt-targeting rule serious. Such back-and-forth fiscal adjustment could leads to overshooting. On the Laffer curve, \( \epsilon^R \) is positive on the left side and negative on the right side, but always decreases as tax rate goes up. The fact that we don’t observe overshooting in reality might be because the economy is on the left side of Laffer curve, but close to the peak.

### 2.2 Complete asset market with capital

I extend the previous model to include capital. Household chooses consumption and working hour according to

\[
\max E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, L_t) \tag{25}
\]
\[
\text{s.t. } c_t + k_t + E_t q_{t+1} b_{t+1} = w_t L_t \left(1 - \tau^L_t\right) + r_t k_{t-1} \left(1 - \tau^k_t\right) + b_t + (1 - \delta) k_{t-1}
\] (26)

Also the allocations satisfy the transversality condition,

\[
E_t \lim_{i \to \infty} Q_{t+i} b_{t+i} = 0
\] (27)

\[
E_t \lim_{i \to \infty} Q_{t+i} R^k_{t+i} k_{t+i-1} = 0
\] (28)

where \(Q_{t+i} = q_{t+1} q_{t+2} \cdots q_{t+i}\) and \(R^k_t = (1 - \tau^k_t) r_t + 1 - \delta\). The interest rate is exogenously determined by the rest of the world. As explained above, \(q_t\) is assumed to be constant and equal to the discount rate \(\beta\). Firm maximize its profit according to

\[
\max f(k_t - 1, L_t) - r_t k_t - 1 - w_t L_t
\] (29)

Finally, government collects uniform tax on labor income and capital, and issue risk-free debt.

\[
d_t + \tau^k_t r_t k_{t-1} + \tau^L_t w_t L_t = R_{t-1} d_{t-1} + g_t
\] (30)

where \(1/R_t = E_t q_{t+1}\). Also, the debt-targeting rule is

\[
\ln \frac{\tau^L_t}{\tau^L} = \gamma^L \ln \frac{S_{t-1}}{S}
\] (31)

\[
\ln \frac{\tau^k_t}{\tau^k} = \gamma^k \ln \frac{S_{t-1}}{S}
\] (32)

where the coefficient \(\gamma^L\) is the fiscal adjustment parameter on income tax, and \(\gamma^k\) on capital tax. \(S^d\) is the debt-GDP ratio, defined as

\[
S_{t-1} = \frac{d_{t-1}}{y_{t-1}}
\] (33)

With explicit utility and production function, this model can be solved analytically. At each period \(t\), income and capital tax rates, \(\tau^L_t\) and \(\tau^k_t\), are predetermined due to the debt-targeting rule. Also, the capital stock \(k_{t-1}\) is predetermined. The predetermined capital stock and income tax rate imply that labor supply \(L_t\) and output \(y_t\) are also predetermined. Moreover, the capital stock for next period \(k_t\) is uniquely determined by the household first-order condition, as the next period labor \(L_{t+1}\) and capital tax rate \(\tau^k_{t+1}\) are predetermined. Therefore dynamic system is nailed down into to a single equation of \(S_t\), debt-GDP ratio at time \(t\), related to the exogenous shock.

\[
S_t + (1 - \alpha) \tau^L (S_{t-1}) + \alpha \tau^k (S_{t-1}) = \frac{1}{\beta} \frac{y (S_{t-2})}{y (S_{t-1})} + \frac{g_t}{y (S_{t-1})}
\] (34)

After log-linearization,

\[
\hat{s}_t + \alpha_1 \hat{s}_{t-1} + \alpha_2 \hat{s}_{t-2} = \frac{g_t}{d} \hat{g}_t
\] (35)

where \(\alpha_1\) and \(\alpha_2\) are coefficients related to model specification parameters. Appendix C includes all the details. The necessary conditions for fiscal sustainability are,

\[
\frac{1}{\beta} > \left(\frac{T_L}{d} + \left(\frac{T}{d} + 1\right) (\epsilon^R_L - 1)\right) \gamma^L + \left(\frac{T_k}{d} + \left(\frac{T}{d} + 1\right) (\epsilon^R_k - 1)\right) \gamma^k
\] (36)
$$\left(\frac{T_L}{d} + \left(\frac{T}{d} + 1 - \frac{1}{\beta}\right) (\epsilon^R - 1)\right) \gamma^L + \left(\frac{T_k}{d} + \left(\frac{T}{d} + 1 - \frac{1}{\beta}\right) (\epsilon^R - 1)\right) \gamma^k > \frac{1}{\beta} - 1 \quad (37)$$

More over, if Equation (36) isn’t met, the dynamic path will fluctuates around the steady states. It says that the more elastic the tax revenue responding to tax rate, the easier to slip into the regime with fluctuations.

3 Optimal debt targeting rule

To address the question of optimal debt targeting rule, I first analyze two small open economies without capital under different bond market structure, i.e. complete or incomplete asset market. Next, I extend the incomplete asset market to include capital, and find that the introduction of capital tax changes the results qualitatively.

3.1 Complete asset market without capital

The model in Section 2.1 is calibrated in such a way that the debt-GDP ratio is about 0.4, the government spending-GDP ratio is 0.2, the trade-balance is 0, and the fraction of time spent working is 0.3. Since the model can be solved nonlinearly, it is straightforward to compute the welfare as the fiscal adjustment parameter \(\gamma\) changes. Figure 1 shows that more rapid fiscal adjustment always improves welfare. This result is robust to different debt-GDP ratio and trade balance-GDP ratio. Since the consumption has been completely insured through the international asset market, their welfare can be improved if agents can enjoy more leisure in more prolonged period. There is a tradeoff. More rapid fiscal adjustment leads to larger tax rate hike at the beginning and people can enjoy more leisure, but it also brings the system back to the steady state faster. On the other hand, slower adjustment lets people enjoy less leisure but in a longer period. In this model, the first effect dominates.

3.2 Incomplete asset market without capital

Now household can only hold risk-free bond, and the interest rate is exogenously determined by the rest of the world. They choose consumption and working hours according to

$$\max_{E^0} \sum_{t=0}^{\infty} \beta^t u(c_t, L_t)$$

s.t. \(c_t + b_t = L_t (1 - \tau_t) + R_t b_{t-1}\) (39)

As the model in Section 2.1, government collects labor income tax and issues risk-free debt to finance spending. Also, they target the debt-GDP ratio. Such a small open economy suffers the well-known non-stationarity problem. I follow Schmitt-Grhoe and Uribe (2003b) and use the debt-elastic interest rate, in the sense that the country risk premium is positively correlated with the aggregate level of foreign debt. Specifically, it is given by

$$R_t = R^* \Psi (d_t - b_t)$$

(40)
where $R^*$ is the foreign interest rate, and $\Psi(\cdot)$ is a country-specific interest rate premium.

In order to compare welfare performance, I use the perturbation method, following Schmitt-Grohe and Uribe (2004). The model specification follows straightly from Section 2.1, except the debt-elastic interest rate. I assume

$$R_t = R^* e^{\psi(d_t - b_t - d + b)}$$

(41)

where $\psi = 0.02$ implies that 1 percent increase of government debt leads to 0.02$d$ percent increase of domestic interest rate, while 1 percent increase of household asset holding leads to 0.02$b$ percent decrease of domestic interest rate. Other than that, the model is calibrated in the same way as the complete asset market scenario. In order to analyze the impact of the economic and fiscal position on the welfare performance, I consider two cases with different government debt-GDP ratio and trade balance-GDP ratio.

<table>
<thead>
<tr>
<th></th>
<th>$\frac{d}{y}$</th>
<th>$\frac{b}{y}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Net lender, low government debt</td>
<td>0.2</td>
<td>-0.02</td>
</tr>
<tr>
<td>Net borrower, high government debt</td>
<td>0.7</td>
<td>0.02</td>
</tr>
</tbody>
</table>

Figure[2] shows how the welfare changes with $\gamma$ when the government has relatively high debt-GDP ratio and the country is a net borrower. Rapid adjustment always benefits household. On the contrary, Figure[3] shows when the government has relatively low debt-GDP and the country is a net lender, slower adjustment improves welfare. The results are intuitive. Under higher tax, leisure becomes less costly relative to consumption and people work less (substitution effect), while lower income encourage people to work more (income effect). Therefore rapid tax adjustment causes dramatic deline of consumption and working hour in short time, while slow adjustment allows relatively higher consumption and less working hour in a prolonged period. If government is in a sound fiscal position, or if household has accumulated high assets, it is less costly to borrow from the international market than otherwise. When higher government spending crowds out private consumption, household can partially lean upon the international market to smooth consumption. Therefore, slower tax adjustment allows people to enjoy more leisure in longer time without compromising much of consumption. On the other hand, life is much harder if the government is in a shaky fiscal position or if the country is a net borrower, as it is costly for them to further borrow from international market. Rapid tax adjustment can make people better off by shortening the painful convergency path.

### 3.3 Incomplete asset market with capital

Household can hold both capital risk-free bond. They chooses consumption and working hour according to

$$\max E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, L_t)$$

(42)

s.t. $c_t + k_t + b_t = w_t L_t \left(1 - \tau_t^L\right) + r_t k_{t-1} \left(1 - \tau_t^k\right) + R_{t-1} b_{t-1} + (1 - \delta) k_{t-1}$

(43)
The interest rate is determined as
\[ R_t = R^* \Psi (d_t - b_t) \] (44)

Firm maximize its profit according to
\[ \max f (k_{t-1}, L_t) - r_t k_{t-1} - w_t L_t \] (45)

Finally, government collects uniform tax on labor income and capital and issue risk-free debt, in addition to target debt-GDP ratio.
\[ d_t + \tau^k_t r_t k_{t-1} + \tau^L_t w_t L_t = R_{t-1} d_{t-1} + g_t \] (46)

Also, the debt-targeting rule is
\[ \ln \frac{\tau^L_t}{\tau^L} = \gamma^L \ln \frac{S_{t-1}}{S} \] (47)
\[ \ln \frac{\tau^k_t}{\tau^k} = \gamma^k \ln \frac{S_{t-1}}{S} \] (48)

The baseline model is calibrated in the way such that labor supply is 0.2 with unit elasticity, and private consumption over GDP ratio is 0.6; capital ratio is 0.35, and capital depreciation rate is 0.36; government spending over GDP ratio is 0.2.

In order to investigate the optimal adjustment parameters, the contour lines of welfare are plotted on the surface of \( \gamma^L \) and \( \gamma^k \). Figure(4-6) shows three different scenarios. In an economy with shaky fiscal condition and holding net foreign debt \( (\frac{d_y}{y} = 0.7, \frac{tb}{y} = 0.02) \), labor tax performs much more efficiently than capital tax. Under modest debt level and balanced current account \( (\frac{d_y}{y} = 0.4, \frac{tb}{y} = 0.0) \), two tax instruments are almost equally efficient. On the other hand, under sound fiscal condition and holding net foreign asset \( (\frac{d_y}{y} = 0.2, \frac{tb}{y} = -0.02) \), capital tax is the better instrument to use.

Capital tax has direct intertemporal effect, in the sense that high capital tax discourages people from accumulating capital and leads to lower output in many future periods. When it is costly to smooth consumption from international market, household can only use capital to smooth consumption and it is not a good idea to heavily tax capital. More rapid adjustment through labor tax can improve the welfare by motivating people sacrifice short-term consumption for the long run. On the contrary, if people can easily borrow from foreigners, capital is less important as an instrument of smoothing consumption. Instead people value more about leisure. Therefore capital tax works more efficient.

### 4 Future work

In pratice, the fiscal rules that many OECD countries currently pursue aren’t as simple as the linear debt targeting rule. Instead, they impose some sort of debt limit rule, like Maastricht Treaty. Government doesn’t need to adjust tax rate unless debt-GDP ratio is approaching the limit. This nonlinear behavior makes it impossible to use the regular approximation method, like the perturbation method applied here. I plan to extend my research into the nonlinear case.
References


Appendix A: Debt rules in selected OECD countries

Fiscal rules in some selected OECD countries (CESifo DICE Report 2/2004):

- **EU countries (Maastricht Treaty (1992))**: 
  - 60 percent of gross government debt-to-GDP ratio norm.
  - 3 percent of GDP ceiling on government net borrowing.

- **New Zealand (Fiscal Responsibility Act (1994))**: 
  - Maintain debt and net worth at prudent levels and run operating surplus on average over a reasonable period of time. The government sets its own numerical targets consistent with these principles.

  - Net debt as a proportion for GDP must be held stable over the business cycle at a prudent level (define so far as net debt below 40 percent of GDP)

- **Australia (Charter of Budget Honesty (1998))**: 
  - The Charter requires the government to spell out objectives and targets but places no constraint on their nature.

- **Denmark (A medium-term fiscal strategy for the period until 2010 (2001))**: 
  - Structural general government surplus of around 2 percent of GDP.
  - A “tax freeze” covering both central and sub-national government (introduced in 2002).
Appendix B: Proof of fiscal sustainability in Section 2

As explained, the model is nailed down into a single equation. It can be log-linearized around the steady states.

\[ \hat{s}_t + \alpha_1 \hat{s}_{t-1} + \alpha_2 \hat{s}_{t-2} = \alpha_3 \hat{g}_t \]  \hspace{1cm} (49)

where

\[ \alpha_1 = \tau L \left( 1 - \frac{1}{\sigma} \frac{\tau}{1 - \tau} \right) - \frac{1}{\beta} - \frac{\gamma}{\sigma} \frac{\tau}{1 - \tau} \]

\[ \alpha_2 = \frac{\gamma}{\beta \sigma} \frac{\tau}{1 - \tau} \]

\[ \alpha_3 = \frac{g}{d} \]  \hspace{1cm} (50)

As explained in the paper, both labor supply and tax rate are tied up to the debt-GDP ratio at previous period through the debt-targeting rule, so the system should be solved backward. Let \( B \) be the lag-operator, then

\[ \hat{s}_t = \frac{\alpha_3 \hat{g}_t}{1 + \alpha_1 B + \alpha_2 B^2} \]  \hspace{1cm} (51)

Define \( \beta_1, \beta_2 = \alpha_2 \) and \( \beta_1 + \beta_2 = -\alpha_1 \), then

\[ \hat{s}_t = \left( \frac{\beta_1}{1 - \beta_1 B} - \frac{\beta_2}{1 - \beta_2 B} \right) \frac{\alpha_3 \hat{g}_t}{\beta_1 - \beta_2} \]  \hspace{1cm} (52)

The system has an explosive path if either \( |\beta_1| \) or \( |\beta_2| \) or both are larger than 1. The necessary and sufficient conditions for the existence of unique equilibrium are

- \( \alpha_1^2 - 4\alpha_2 > 0 \)
- \( \alpha_1 < 0 \) and \( -\frac{\alpha_1 + \sqrt{\alpha_1^2 - 4\alpha_2}}{2} < 1 \); or \( \alpha_1 > 0 \) and \( -\frac{\alpha_1 - \sqrt{\alpha_1^2 - 4\alpha_2}}{2} > -1 \)

The conditions are equivalent to

- \( \alpha_1^2 - 4\alpha_2 > 0 \)
- \( 0 > \alpha_1 > -2 \) and \( \alpha_1 + \alpha_2 + 1 > 0 \); or \( 2 > \alpha_1 > 0 \) and \( 1 - \alpha_1 + \alpha_2 > 0 \)

If the above conditions are met, the system can be solved as

\[ \hat{s}_t = \sum_{i=0}^{t-1} \frac{\beta_1^{i+1} - \beta_2^{i+1}}{\beta_1 - \beta_2} - \alpha_3 \hat{g}_{t-i} \]  \hspace{1cm} (53)

Assume the spending shock follows

\[ \hat{g}_t = \rho \hat{g}_{t-1} + \eta_t \]  \hspace{1cm} (54)

After one-time shock \( \eta_0 \), \( \hat{s}_t \) can be simplified as

\[ \hat{s}_t = \frac{\alpha_3}{\beta_1 - \beta_2} \left( \frac{\beta_1}{1 - \beta_1 \rho} - \frac{\beta_2}{1 - \beta_2 \rho} \right) \hat{g}_0 - \frac{\alpha_3}{\beta_1 - \beta_2} \left( \frac{\beta_1^t + \beta_2^t}{1 - \beta_1 \rho} - \frac{\beta_1^t + \beta_2^t}{1 - \beta_2 \rho} \right) \rho^{t+1} \hat{g}_0 \]  \hspace{1cm} (55)
Note that $\alpha_2$ is always positive, it implies the $\beta_1$ and $\beta_2$ are either both positive or both negative, depending on $\alpha_1$. If $\alpha_1$ is negative, the two roots are positive, and the path of $\tau_t$ steadily converge back to the steady state. On the other hand, with positive $\alpha_1$, $\beta_1$ and $\beta_2$ are negative, and $\tau_t$ fluctuates around the steady states before it dies out eventually.

It can be shown that the necessary and sufficient conditions for the existence of unique and stable equilibrium are

$$\left( \left( \frac{1}{\beta} - 1 \right) (1 - \epsilon^R) + \frac{T}{d} \epsilon^R \right) \gamma > \frac{1}{\beta} - 1 \tag{56}$$

$$\frac{1}{\beta} > \left( \frac{T}{d} \epsilon^R + \epsilon^R - 1 \right) \gamma > \frac{1}{\beta} - 2 \tag{57}$$

$$\left( \sqrt{\frac{T}{d} \epsilon^R - \sqrt{1 - \epsilon^R}} \right) \sqrt{\gamma} > \sqrt{\frac{T}{\beta}} \tag{58}$$

$$\epsilon^R > 0 \tag{59}$$

More specifically, the path starts to fluctuate if

$$\left( \frac{T}{d} \epsilon^R + \epsilon^R - 1 \right) \gamma > \frac{1}{\beta} \tag{60}$$

where $T = \tau L$ is the tax revenue at steady state, $\epsilon^R = \frac{\bar{p}}{\bar{r}}$ is the uncompensated elasticity between tax revenue and tax rate. The larger the elasticity $\epsilon^R$, the easier to reach the regime with fluctuation.
Appendix C: Proof of fiscal sustainability in Section 4

In the economy with capital and complete asset market, the first order conditions for household are straightforward,

\[ u_c(t) q_{t+1} = \beta u_c(t+1) \]  \hspace{1cm} (61)

\[ -\frac{u_L(t)}{u_c(t)} = (1 - \tau_t^L) w_t \]  \hspace{1cm} (62)

\[ u_c(t) = E_t \beta u_c(t+1) \left( (1 - \tau_{t+1}^k) r_{t+1} + 1 - \delta \right) \]  \hspace{1cm} (63)

The firm’s maximization problem implies that it implies that

\[ r_t = f_k(k_{t-1}, L_t) \]  \hspace{1cm} (64)

\[ w_t = f_L(k_{t-1}, L_t) \]  \hspace{1cm} (65)

Assume the following functional forms,

\[ u(c, L) = \ln c - \chi L^{1+\sigma} \]  \hspace{1cm} (66)

\[ f(k, L) = k^\alpha L^{1-\alpha} \]  \hspace{1cm} (67)

The dynamic system can be written as

\[ L_t^{\sigma+\alpha} = \frac{1 - \alpha}{\chi c} (1 - \tau_t^L) k_{t-1}^\alpha \]  \hspace{1cm} (68)

\[ y_t = k_{t-1}^\alpha L_t^{1-\alpha} \]  \hspace{1cm} (69)

\[ E_t \left( (1 - \tau_{t+1}^k) \alpha k_t^{\alpha-1} L_t^{1-\alpha} \right) = \frac{1}{\beta} + \sigma - 1 \]  \hspace{1cm} (70)

\[ d_t = \frac{1}{\beta} d_{t-1} + g_t - (1 - \alpha) y_t \tau_t^L - \alpha y_t \tau_t^k \]  \hspace{1cm} (71)

\[ \tau_t^L = \left( \frac{S_{t-1}^d}{S_t^d} \right)^{\gamma_L} \]  \hspace{1cm} (72)

\[ \tau_t^k = \left( \frac{S_{t-1}^d}{S_t^d} \right)^{\gamma_k} \]  \hspace{1cm} (73)

After some substitutions, the path of capital can be shown as

\[ k_t = a_4 \left( 1 - a_2 \left( \frac{d_t}{y_t} \right)^{\gamma_k} \right)^{\frac{\sigma+\alpha}{\sigma(1-\alpha)}} \left( 1 - a_3 \left( \frac{d_t}{y_t} \right)^{\gamma_L} \right)^{\frac{1}{\gamma}} \]  \hspace{1cm} (74)

where

\[ a_4 = \frac{\alpha \left( \frac{1-\alpha}{\chi c} \frac{1}{\gamma} \right)^{\frac{\sigma+\alpha}{\sigma(1-\alpha)}} \frac{\sigma+\alpha}{\pi(1-\alpha)}}{1/\beta + \delta - 1} \]  \hspace{1cm} (75)
\( a_2 = \frac{\gamma^k}{(Sd)^{\gamma^k}} \)  
(76)

\( a_3 = \frac{\gamma^L}{(Sd)^{\gamma^L}} \)  
(77)

Substitute it into the household first-order condition for labor,

\[
L_t = a_5 \left(1 - a_2 \left(\frac{d_t}{y_t}\right)^{\gamma^k}\right)^{\frac{\alpha}{(\alpha + 1)\beta}} \left(1 - a_3 \left(\frac{d_t}{y_t}\right)^{\gamma^L}\right)^{\frac{1}{\beta}}
\]  
(78)

where

\[
a_5 = \left(\frac{\alpha}{1/\beta + \delta - 1}\right)^{\frac{1}{(\alpha + 1)\beta}} \left(\frac{1 - \alpha}{\chi c}\right)^{\frac{1}{\beta}}
\]  
(79)

Therefore

\[
y_t = a_6 \left(1 - a_2 \left(\frac{d_t}{y_t}\right)^{\gamma^k}\right)^{\frac{\alpha(1+\sigma)}{(\alpha + 1)\beta}} \left(1 - a_3 \left(\frac{d_t}{y_t}\right)^{\gamma^L}\right)^{\frac{1}{\beta}}
\]  
(80)

where \( a_6 = a_4 a_5^{1-\alpha} \).

The government budget constraint can be rewritten as

\[
\frac{d_t}{y_t} = \frac{1}{\beta} \frac{d_{t-1} y_{t-1}}{y_t} + \frac{g_t}{y_t} - (1 - \alpha) \tau^L_t - \alpha \tau^k_t
\]  
(81)

Define \( S_t = \frac{d_t}{y_t} \). Substitute out \( y_t, y_{t-1}, \tau^L_t \) and \( \tau^k_t \). The above equation shows how the path of \( S_t \) relates to exogenous shock \( g_t \). It can be log-linearized around the steady state.

\[
\hat{s}_t + \alpha_1 \hat{s}_{t-1} + \alpha_2 \hat{s}_{t-2} = \frac{g}{d} \hat{g}_t
\]  
(82)

where

\[
\alpha_1 = -a_2 \left(\frac{1}{\beta} + \frac{g}{d}\right) - \frac{1}{\beta} + (1 - \alpha) \frac{a_3}{S} \gamma^L S^{\gamma^L} + \frac{a_2 \alpha}{S} \gamma^k S^{\gamma^k}
\]  
(83)

\[
\alpha_2 = \frac{1}{\beta} \left(\frac{\alpha}{1 - \alpha} \frac{1 + \sigma}{1 - a_2} \frac{S^{\gamma^k}}{S^{\gamma^k}} + \frac{1}{\sigma} \frac{a_3 \gamma^L S^{\gamma^L}}{1 - a_3 S^{\gamma^L}}\right)
\]  
(84)

Follow the same strategy, \( S_t \) can be solved backward. The necessary condition for a unique equilibrium are

- \( \alpha_1^2 - 4\alpha_2 > 0 \)
- \( \alpha_1 < 0 \) and \( -\frac{\alpha_1 + \sqrt{\alpha_1^2 - 4\alpha_2}}{2} < 1 \); or \( \alpha_1 > 0 \) and \( -\frac{\alpha_1 - \sqrt{\alpha_1^2 - 4\alpha_2}}{2} > -1 \)

Or equivalent to

- \( \frac{1}{\beta} > \lambda_1 \gamma^L + \lambda_2 \gamma^k > \frac{1}{\beta} - 2 \);
\[ \lambda_3 \gamma^L + \lambda_4 \gamma^k > \frac{1}{\beta} - 1 \]

where

\[
\lambda_1 = \frac{T_L}{d} + \left( \frac{T}{d} + 1 \right) (\epsilon^R_L - 1) \tag{85}
\]

\[
\lambda_2 = \frac{T_k}{d} + \left( \frac{T}{d} + 1 \right) (\epsilon^R_k - 1) \tag{86}
\]

\[
\lambda_3 = \frac{T_L}{d} + \left( \frac{T}{d} + 1 - \frac{1}{\beta} \right) (\epsilon^R_L - 1) \tag{87}
\]

\[
\lambda_4 = \frac{T_k}{d} + \left( \frac{T}{d} + 1 - \frac{1}{\beta} \right) (\epsilon^R_k - 1) \tag{88}
\]

Note the total tax revenue \( T \) is the sum of revenue from labor tax \( T_L \) and from capital tax \( T_k \). \( \epsilon^R_L = \frac{\tau_L}{T_L} \) is the elasticity of labor tax revenue, while \( \epsilon^R_k = \frac{\tau_k}{T_k} \) is the elasticity of capital tax revenue.

In another word, if \( \lambda_1 \gamma^L + \lambda_2 \gamma^k \) is larger than the \( 1/\beta \), there exists fluctuations.
Appendix D: Graphs

Figure 1: Welfare under complete asset market without capital

\[ \pi_{10^3} \text{ Complete Asset Market without capital} \]
Figure 2: Welfare under incomplete asset market without capital (large debt-GDP)

Figure 3: Welfare under incomplete asset market without capital (small debt-GDP)
Figure 4: Welfare contour lines under incomplete asset market with capital (large debt-GDP)
Figure 5: Welfare contour lines under incomplete asset market with capital (modest debt-GDP)
Figure 6: Welfare contour lines under incomplete asset market with capital (small debt-GDP)