Outline of the presentation

1. A quick revision of mean-variance frontiers for arbitrage portfolios
2. Unrestricted estimation of mean-variance frontiers
   - Frontier estimation
   - Statistical properties
   - Confidence regions
3. Estimation of mean-variance frontiers imposing spanning restrictions
   - Frontier estimation
   - Mean-variance efficiency tests
4. Estimation of mean-variance frontiers imposing asset pricing restrictions
   - Frontier estimation
   - Linear factor pricing tests
1. A quick revision of mean-variance frontiers for arbitrage portfolios

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Let $\mathbf{r}$ denote a vector of excess returns on $n$ assets. These portfolios span the space of zero-cost portfolios, $\mathcal{A}$. We saw before that the cost representing portfolios in this space are 0. We also saw that the mean representing portfolios are:

$$
\alpha^\circ = E(\mathbf{r}')[E(\mathbf{r}\mathbf{r}')]^{-1}\mathbf{r},
$$

$$
\delta^\circ = E(\mathbf{r}')[V(\mathbf{r})]^{-1}\mathbf{r} = [1 - E(\alpha^\circ)]^{-1}\alpha^\circ.
$$
The elements of the SMVF calculated from $r$ are:

$$m_{MV}^{c}(c) = \frac{c}{1 - E(a^\circ)} (1 - a^\circ) = c\{1 - [\bar{\delta}^\circ - E(\bar{\delta}^\circ)]\},$$

and their variance will be

$$\sigma^2(c) = c^2 \frac{E(a^\circ)}{1 - E(a^\circ)} = c^2 E(\bar{\delta}^\circ).$$

Similarly, the elements of the AMVF are:

$$r_{MV}^{\mu}(\mu) = \mu \frac{1}{E(a^\circ)} a^\circ = \mu \frac{1}{E(\bar{\delta}^\circ)} \bar{\delta}^\circ,$$

and their variance is

$$V[r_{MV}^{\mu}(\mu)] = \frac{1 - E(a^\circ)}{E(a^\circ)} \mu^2 = \frac{1}{E(\bar{\delta}^\circ)} \mu^2.$$

I will focus the presentation on the SMVF, but the graphs will show the AMVF.
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Uncentred moment conditions

- The SMVF is linear in the single unknown parameter

\[ \theta = E(\delta^\circ) = [1 - E(a^\circ)]^{-1} E(a^\circ), \]

which can be interpreted as the maximum (square) Sharpe ratio attainable.

- In turn, the AMVF frontier is linear in its reciprocal, \( \theta^{-1} \).

- Given a vector of \( n \) excess returns \( r \), we can estimate both frontiers from the following exactly identified system of \( n + 1 \) moment conditions:

\[ E \left[ \begin{array}{c} rr' \phi^\circ - r \\ r' \phi^\circ - \theta/(1 + \theta) \end{array} \right] = 0, \]

where \( \theta/(1 + \theta) \) identifies \( E(a^\circ) = E(a^{\circ 2}) \) and \( \phi^\circ \) the portfolio weights of this uncentred mean representing portfolio.
Centred moment conditions

- Alternatively, we could work with the analogous $n + 1$ moment conditions for the centred representing portfolio $\delta$

$$E \left[ \begin{array}{c} r (r' \varphi - \theta) - r' \\ r' \varphi - \theta \end{array} \right] = 0,$$

where $\theta$ identifies $E (\delta^\circ) = V (\delta^\circ)$ and $\varphi^\circ$ the corresponding portfolio weights.

- Centred and uncentred moment conditions are equivalent in the sense that they provide the same numerical estimate of $\theta$ and the same standard error through the Delta method.
Under standard regularity conditions, the resulting GMM estimator of $\theta$ will converge in probability to its true value, and the same is true of the weights of the mean representing portfolios.

Therefore, the GMM estimators of $V[m^{MV}(c)]$ and $V[r^{MV}(\mu)]$ will also converge in probability to their population counterparts for fixed $c$ and $\mu$.

We can also show that GMM estimators of the entire SMVF and AMVF will converge uniformly to their population analogues over any finite range.

For example, in the case of the SMVF frontier

$$\sup_{c \in [\underline{c}, \overline{c}]} |c^2 \hat{\theta} - c^2 \theta| = \left( \sup_{c \in [\underline{c}, \overline{c}]} c^2 \right) |\hat{\theta} - \theta| = o_p(1).$$
Sampling variability

- Despite the uniform consistency, though, the SMVF and AMVF frontiers are subject to substantial sample variability.
- To emphasise the importance of sampling uncertainty in this context, we have conducted the following simulation experiment.
- We have assumed that investors have access to six arbitrage portfolios, whose excess returns roughly replicate the distribution of the 6 Fama and French portfolios formed on size and book-to-market.
- Then we simulate 40 years of monthly data many times, and compute the mean-variance frontiers.
Sampling variability

Sampling distribution of Sharpe ratio
Confidence regions

- Suppose we are interested in a joint confidence region at $c_1$ and $c_2$.
- We need the joint asymptotic distribution of the estimators of the variance of the SMVF elements corresponding to those two values:
  \[
  \sqrt{T} \left[ \begin{array}{c} \hat{\sigma}^2 (c_1) - \sigma^2 (c_1) \\ \hat{\sigma}^2 (c_2) - \sigma^2 (c_2) \end{array} \right] \xrightarrow{d} N \left[ 0, \nu \begin{pmatrix} \nu_1 & \nu_2 \\ \nu_2 & \nu_1 \end{pmatrix} \right],
  \]
  where $\nu$ denotes the asymptotic variance of $\sqrt{T} (\hat{\theta} - \theta)$.
- The singularity of this distribution implies that the joint confidence region for $\sigma^2 (c_1)$ and $\sigma^2 (c_2)$ will be a unidimensional interval.
- The same is true regardless of the number of values of $c$.
- In the two-point case, the joint confidence region in $[\sigma^2 (c_1), \sigma^2 (c_2)]$ space will be given by an interval on a straight line:
  \[
  \left\{ c_1^2 \left[ \hat{\sigma}^2 (c_1) - \sigma^2 (c_1) \right] + c_2^2 \left[ \hat{\sigma}^2 (c_2) - \sigma^2 (c_2) \right] \right\}^2 \leq \frac{\nu \left( \frac{c_1^4 + c_2^4}{T} \right)^2}{\chi^2_{1;1-\alpha}},
  \]
  where $\chi^2_{1;1-\alpha}$ is the $1 - \alpha$ quantile of a $\chi^2_1$. 

Confidence regions

SDFs for x

Variance of c=0.90 vs Variance of c=0.95

Graph showing a linear relationship between variance for c=0.90 and c=0.95.
Confidence regions

- Although we cannot represent those confidence intervals in more than three dimensions, we can represent them in \([c, \sigma^2(c)]\) space by finding the projection of the joint interval onto the \(\sigma^2(c_1)\) and \(\sigma^2(c_2)\) axes.
- The projection over \(\sigma^2(c_1)\) will be given by

\[
\left[\hat{\sigma}^2(c_1) - \sigma^2(c_1)\right]^2 \leq \frac{uc_1^4}{T} \chi_{1;1-\alpha}^2
\]

irrespective of the value of \(c_2\).
- Therefore, the representation of the \(1 - \alpha\) uniform confidence bands on \([c, \sigma^2(c)]\) space will be

\[
c^2 \left[\hat{\theta} \pm \sqrt{\frac{u}{T}} \chi_{1;1-\alpha}^2\right].
\]

- Not surprisingly, the width of these bands increases with \(c, u\) and \(\alpha\), and decreases with the sample size.
Confidence regions

Mean vs. Standard deviation

XS for x

Enrique Sentana
MV Frontiers for Zero-Cost Portfolios
The region thus generated will also have the right coverage in $[c, \sigma^2(c)]$ space because

$$
\lim_{T \to \infty} \text{Pr} \left\{ \begin{aligned}
&c^2 \left[ \hat{\theta} + \sqrt{\frac{T}{\nu}} \chi_{1;1-\alpha}^2 \right] \leq c^2 E(\delta^o) \\
&\leq c^2 \left[ \hat{\theta} + \sqrt{\frac{T}{\nu}} \chi_{1;1-\alpha}^2 \right] \quad \forall c \in [\underline{c}, \bar{c}]
\end{aligned} \right\} = 1 - \alpha.
$$

This result follows from the fact that the upper and lower bounds correspond to the maximum and minimum values of $\sigma^2(c) = c^2 \theta$ that can be achieved within the $1 - \alpha$ confidence interval:

$$
T \left( \frac{\hat{\theta} - \theta}{\nu} \right)^2 \leq \chi_{1;1-\alpha}^2.
$$

In fact, the uniform confidence bands also coincide with the pointwise confidence bands for $\sigma^2(c)$ due to the presence of a single estimated parameter.
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Spanning restrictions

- Partition $r$ into two sets of portfolios $r_1$ and $r_2$ of dimensions $n_1$ and $n_2$, respectively, with $n = n_1 + n_2$, so that $r' = (r'_1, r'_2)$.
- Theoretical or empirical considerations may suggest that the addition of $r_2$ should not improve the investment opportunity set of investors.
- Equivalently, we may believe that the inclusion of $r_2$ would not tighten the bounds on admissible SDFs.
- In both cases, we say that $r_1$ spans the MV frontiers generated from $r_1$ and $r_2$.
- Under the null hypothesis, there will be one pair of MV frontiers.
- Under the alternative, there will be two: those generated from $r_1$ alone, and those generated from $r$, which will only touch at the origin.
- We already know how to estimate those unrestricted frontiers.
- How can we efficiently estimate the unique frontiers under the null?
- Despite first appearances, the result will be generally different from the unrestricted frontiers estimated on the basis of $r_1$ alone.
Efficient estimation of curves is a somewhat unusual concept in econometrics.

Given two alternative estimators of a given curve, we could say that one is more efficient than the other if loosely speaking some efficiency gains accrue in estimating any arbitrary vector of points on the curve.

Alternatively, we could say that one curve estimator is more efficient than another curve estimator if their efficiency ranking is preserved for any linear functional.

Since in our case the entire SMVF and AMVF depend exclusively on a single parameter estimator, both concepts trivially imply that more efficient “curve estimators” of the frontiers will be obtained by using more efficient estimators of $\theta$. 
The null hypothesis of spanning imposed on the weights on the uncentred mean representing portfolio gives rise to the overidentified system

\[
E \left[ \begin{pmatrix} r_1 \\ r_2 \\ 1 \end{pmatrix} r_1' \phi_1^o - \begin{pmatrix} r_1 \\ r_2 \\ \theta/(1 + \theta) \end{pmatrix} \right] = 0.
\]

The optimal GMM estimator of \( \theta \) so obtained will be generally more efficient than the corresponding estimator obtain from the unrestricted moment conditions as long as the equality restriction \( \phi_2^o = 0 \) holds.

Moreover, this estimator will be generally more efficient than the one obtained from the just identified \( n_1 + 1 \) moment conditions

\[
E \left[ \begin{pmatrix} r_1 \\ 1 \end{pmatrix} r_1' \phi_1^o - \begin{pmatrix} r_1 \\ \theta/(1 + \theta) \end{pmatrix} \right] = 0.
\]

Enrique Sentana
MV Frontiers for Zero-Cost Portfolios
A spherically symmetric random vector of dimension \(n\), \(\varepsilon^o\), is characterised as \(\varepsilon^o = e_t u\), where \(u\) is uniformly distributed on the unit sphere surface in \(\mathbb{R}^n\), and \(e\) is a non-negative random variable independent of \(u\), whose distribution determines the distribution of \(\varepsilon^o\).

The variables \(e\) and \(u\) are referred to as the generating variate and the uniform base of the spherical distribution. Assuming that \(E(e^2) < \infty\), we can standardize \(\varepsilon^o\) by setting \(E(e^2) = n\), so that \(E(\varepsilon^o) = 0\), \(V(\varepsilon^o) = I_n\).
Elliptical Distributions: Examples

- **Gaussian distribution**: $\epsilon^* = \sqrt{\varsigma} u$ where $\varsigma$ is a $\chi^2$ random variable with $n$ degrees of freedom.

- **Student $t$ distribution**: $\epsilon^* = \sqrt{\nu_0 - 2 \times \sqrt{\zeta/\xi}} u$ where $\zeta$ is a $\chi^2$ random variable with $n$ degrees of freedom, and $\xi$ is an independent Gamma variate with mean $\nu_0$ and variance $2\nu_0$.

- **Kotz distribution**: $\epsilon^* = \sqrt{\varsigma} u$ where $\varsigma$ is a gamma random variable with mean $n$ and variance $n[\nu + 2\kappa_0 + 2]$.  

- **Discrete scale mixture of normals distribution**: $\epsilon^* = \sqrt{\varsigma} u$ where

\[
\varsigma = \frac{s + (1 - s)\kappa_0}{\pi_0 + (1 - \pi_0)\kappa_0} \zeta
\]

where $\zeta$ is a $\chi^2$ random variable with $n$ degrees of freedom and $s$ is an independent Bernoulli variate with $P(s = 1) = \pi_0$ and $\kappa_0$ is the variance ratio of the two components.
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where $\zeta$ is a $\chi^2$ random variable with $n$ degrees of freedom and $s$ is an independent Bernoulli variate with $P(s = 1) = \pi_0$ and $\kappa_0$ is the variance ratio of the two components.
Elliptical Distributions: Densities and Contours

Gaussian

Student

Kotz
It turns out that if the distribution of $r$ is elliptical, there are no efficiency gains in estimating $E(a^\circ)$.

Specifically, the asymptotic variances of the unrestricted GMM estimators of this parameter based on $r_1$ and $r$ will coincide, and they will be equal to the asymptotic variance of the restricted optimal GMM estimator based on $r$ that imposes the spanning constraint that $\phi_2^\circ$ is 0.

On the other hand, the asymptotic variance of the unrestricted GMM estimator of $\phi_1$ based on $r$ will be typically larger than the asymptotic variance of the unrestricted estimator based on $r_1$, which will coincide with the the asymptotic variance of the restricted optimal GMM estimator based on $r$ that imposes the spanning constraint $\phi_2^\circ = 0$.

Obviously, the same is trivially true of $\phi_2$. 
Efficient estimation imposing spanning restrictions

![Graph showing mean vs standard deviation with lines indicating true, unrestricted, and restricted scenarios.](image-url)
Confidence regions

XS – unrestricted and restricted (blue and red)
These efficiency gains come at a cost: if the spanning restrictions are wrong, then the “efficient” frontier estimator will be inconsistent.

There is a huge literature on testing these restrictions, which usually comes under the heading of mean-variance efficiency tests.

The advantage of our GMM set-up is that we can readily use the overidentification test of the moment conditions that impose the constraint to test for spanning, since it coincides with the distance metric test of the null hypothesis $H_0 : \phi_2^o = 0$

This test will have a limiting chi-square distribution with $n_2$ degrees of freedom under the null.

An analogous test could be based on the moment conditions that define the centred mean representing portfolio.

Since $[E(rr')]^{-1}E(r) = \{1 + E(r') [V(r)]^{-1}E(r)\}^{-1} [V(r)]^{-1} E(r)$ by virtue of the Woodbury formula, $\phi_2^o$ and $\varphi_2^o$ will be proportional to each other, and the null hypotheses are equivalent.
The most popular mean-variance efficiency tests by far are the regression-based tests considered by Gibbons, Ross and Shanken (1989), and robustified by MacKinlay and Richardson (1991). Their test would correspond to the overidentification test of the \( n_2(n_1 + 1) \) moment conditions

\[
E \left[ \left( \begin{array}{c} 1 \\ r_1 \end{array} \right) \otimes (r_2 - Br_1) \right] = 0.
\]

Peñaranda and Sentana (2010b) show that all three approaches (namely, uncentred and centred representing portfolios and regression) are numerically equivalent when implemented by single-step methods such as CU-GMM. This fact also implies that the three approaches will be asymptotically equivalent when implemented by two-stage or iterated GMM, even though they will not be numerically equivalent in that case.
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We could also consider using an asset pricing model to reduce the sampling uncertainty in the construction of portfolio frontiers.

The standard approach in empirical finance is to model $m$ as an affine transformation of some $k \leq n$ observable risk factors $f$, even though this ignores that $m$ must be positive with probability 1 to avoid arbitrage opportunities.

We can express the pricing equation as

$$E \left[ (\lambda_0 - \lambda' f) r \right] = 0$$

for some real numbers $(\lambda_0, \lambda')'$.

We can in fact understand the spanning restrictions discussed in the previous section as imposing a linear factor pricing model in which the pricing factors $f$ coincide with some excess returns $r_1$, as in the CAPM or the Fama and French (1993) model.

In general, though, $f$ does not have to be a subset of $r$. 
Estimation imposing asset pricing restrictions

- Although \( r \) only contains assets with 0 cost, which leaves the scale and sign of \( m \) undetermined, we require a scale normalisation to rule out the trivial solution \((\lambda_0, \lambda')' = (0, 0')'.\)

- For example, we could choose the popular asymmetric normalisations \( \lambda_0 = 1 \) or \( E(m) = \lambda_0 - \lambda'E(f) = 1. \)

- For simplicity, we will follow the former normalisation in our exposition, although as shown by Peñaaranda and Sentana (2010b), normalisation is inconsequential for single-step GMM methods.

- Assuming that \( f \) and \( r \) do not share any common elements, we get

\[
E \begin{bmatrix}
    r (1 - f'\lambda) \\
    rr'\phi^\circ - r \\
    r'\phi^\circ - \theta/(1 + \theta)
\end{bmatrix} = 0,
\]

where the unknown parameters are \((\lambda', \phi'^\circ, \theta).\)

- In this way, we obtain more efficient estimators of the mean-variance frontiers that exploit the pricing equations.
The overidentifying restriction test of the previous moment conditions yields a valid asset pricing test, whose asymptotic distribution will be a chi-square with \( n - k \) degrees of freedom.

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This fact also implies that the three approaches will be asymptotically equivalent when implemented by two-stage or iterated GMM, even though they will not be numerically equivalent in that case.
References

- Peñaranda, F. and Sentana, E. (2011): “Inference about portfolio and stochastic discount factor mean variance frontiers”, work in progress, CEMFI.