Introduction to Portfolio and Stochastic Discount Factor
Mean Variance Frontiers

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Outline of the presentation

1. Elements of financial decision theory under uncertainty
2. Digression on Hilbert spaces, projections and functionals
3. Portfolio selection
4. Mean-variance frontiers
5. The effects of considering additional assets
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Elements of financial decision theory under uncertainty

- Single consumption good
- Discrete-state setting
- States of nature (outcomes or elementary events): \( \omega_s, s = 1, \ldots, S \)
- Sample space: \( \Omega \)
- Event space: \( \mathcal{F} (\sigma\text{-algebra}) \)
- Probability measure: \( \Pi : \pi_s \geq 0, \sum_{s=1}^{S} \pi_s = 1 \)
- Risky asset payoffs: \( x_{is} (i = 1, \ldots, N) \)
- Safe asset: \( x_{0s} = 1 \ \forall s \)
Elements of financial decision theory under uncertainty

- Financial engineering: Combine existing assets to create new ones
- Fixed weight portfolios $p_s = w_ix_is + w_jx_js$, $w_i, w_j \in \mathbb{R}$
- Contingent claims (Arrow-Debreu securities): $ad_{rs} = I(r = s)$, $(r = 1, \ldots, S)$
- Market completeness
- Felicity function $u(x_is; s, \theta)$
- If state independent, then we can regard asset payoffs as random variables $x_j$ defined on the underlying probability space $(\Omega, \mathcal{F}, \Pi)$
- Preference ordering: $x_i \succ x_j$, $x_i \prec x_j$ or $x_i \sim x_j$
- Expected utility rule: $x_i > x_j \iff E[u(x_i; \theta)] - E[u(x_j; \theta)] > 0$
Properties of expected utility

1. Separable in outcomes
2. Linear in probabilities:
   \[ E[u(x_i; \theta)] = \sum_{s=1}^{S} \pi_s u(x_{is}; \theta) = \int u(x; \theta) dF_{x_i}(x) \]
3. Defined up to affine transformations
4. Risk aversion: \( E[u(x_i; \theta)] < u[E(x_i); \theta] \) (concavity of \( u(.) \))

Some commonly used utility functions:

1. Linear utility (risk neutrality): \( u(x_{is}; \theta) = x_{is} \)
2. Quadratic utility: \( u(x_{is}; \theta) = x_{is} + \alpha x_{is}^2 \)
3. Exponential utility: \( u(x_{is}; \theta) = - \exp(-\gamma x_{is}) \)
4. Power utility: \( u(x_{is}; \theta) = (1 - \phi)^{-1} \left( x_{is}^{1-\phi} - 1 \right) \)
5. Logarithmic utility: \( u(x_{is}; \theta) = \ln x_{is} \) (\( \lim_{\phi \to 1} \))
Elements of financial decision theory under uncertainty

- Asset prices: $C(x_i)$
- Law of one price: $C(w_ix_i + w_jx_j) = w_iC(x_i) + w_jC(x_j)$ (Linear pricing)
- Under complete markets $C(x_i) = \sum_{s=1}^{S} x_{is}C(ad_s)$
- Stochastic discount factor: $m$

\[
E(x_im) = \sum_{s=1}^{S} x_{is}m_s\pi_s = C(x_i)
\]

- $E(x_0m) = E(m) = C(1)$
- $C(x_i) = E(x_i)E(m) + \text{cov}(x_i, m)$
- Under complete markets $m_s = C(ad_s)/\pi_s$ for those $s$ for which $\pi_s > 0$
Absence of arbitrage opportunities $m_s > 0$ for all $s$ for which $\pi_s > 0$

Also

$$\frac{C'(ad_s)}{C'(ad_r)} = \frac{\pi_s \cdot u'(x_s^o; \theta)}{\pi_r \cdot u'(x_r^o; \theta)}$$

Under risk neutrality $C'(ad_s)/C'(ad_r) = \pi_s/\pi_r \Rightarrow m_s = k \ \forall s$
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A Banach space is a complete normed linear space.

A Hilbert space is a Banach space whose norm arises from an inner product.

Example: \( L^2 : \{ x : E(x^2) < \infty \} \)

Mean square inner product: \( E(yx) \), \( x, y \in L^2 \)

Mean square norm: \( \sqrt{E(x^2)} \), \( x \in L^2 \)

\( x = y \iff E(x - y)^2 = 0 \)

Any closed linear subspace of \( L^2 \), \( H \) say, is also a Hilbert space.

If \( V(h) > 0 \ \forall h \in H \), then \( H \) is also a Hilbert space with respect to the covariance inner product, \( \text{cov}(h_1, h_2) \) and the associated standard deviation norm \( V^{1/2}(h) \).
Least squares projections: 

\[ P(x|H) \in H \text{ is such that } E [x - P(x|H)]^2 \leq E (x - h)^2 \forall h \in H \]

Properties of projections:

1. Orthogonality condition: \( E \{[x - P(x|H)] h\} = 0 \forall h \in H \)
2. Linearity: \( P(ax + by|H) = aP(x|H) + bP(y|H) \), \( a, b \in \mathbb{R} \)
3. Updating property: Let \( G \) be another closed linear subspace of \( L^2 \) such that \( H \subseteq G \). Then \( P(x|G) = P(x|H) + P(x|V) \), where \( V = \{v : v = g - P(g|H), \text{ for some } g \in G\} \), so that \( G = H \oplus V \)
4. Projection theorem: \( x = P(x|H) + [x - P(x|H)] \) (unique)
5. Law of iterated projections: \( P(x|H) = P [P(x|G)|H] \)
Examples:

1. **Conditional expectations:** If $G$ contains all possible Borel-measurable functions of $z$ with bounded second moments, then $P(x|G) = E(x|z)$

2. **Linear projections:** If $H = \langle y, z \rangle$, where $\langle y, z \rangle = \{\alpha y + \beta z, \alpha, \beta \in \mathbb{R}\}$ is the linear span of $y$ and $z$, then

   $$P(x|H) = \begin{bmatrix} E(xy) \\ E(xz) \end{bmatrix} \begin{bmatrix} E(y^2) & E(yz) \\ E(yz) & E(z^2) \end{bmatrix}^{-1} \begin{bmatrix} y \\ z \end{bmatrix}$$

3. $P(x|\langle 1 \rangle) = E(x|1) = E(x)$
Linear functional
\[ \pi() : L^2 \to \mathbb{R} \text{ is linear on } H \subseteq L^2 \text{ iff } \]
\[ \pi(a_1 h_1 + a_2 h_2) = a_1 \pi(h_1) + a_2 \pi(h_2) \forall a_1, a_2 \in \mathbb{R} \text{ and } \forall h_1, h_2 \in H. \]

Continuous functional:
\[ \pi() : L^2 \to \mathbb{R} \text{ is continuous on } H \subseteq L^2 \text{ iff } \]
\[ \lim_{n \to \infty} E(h_n - h)^2 = 0 \Rightarrow \lim_{n \to \infty} \pi(h_n) = \pi(h) \text{ for } \{h_n\}, h \in H. \]

Riesz representation theorem:
If \( \pi() \) is a continuous linear functional mapping \( H \) on \( \mathbb{R} \), then
\[ \exists! h^\# \in H \text{ such that } \pi(h) = E(hh^\#) \forall h \in H. \]
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Portfolio selection

- Risky assets: \( \mathbf{x} = (x_1, \ldots, x_N)' \), \( N < \infty \).
- They could be primitive assets, such as stocks and bonds, or mutual funds managed according to some active portfolio strategy.
- Assume \( tr[E(\mathbf{x}\mathbf{x}')]<\infty \), so \( x_i \in L^2 \)
- \( |E(\mathbf{x}\mathbf{x}')| \neq 0 \) (no redundant assets) \( \Rightarrow \) Law of one price
- \( V(\mathbf{x}) = E(\mathbf{x}\mathbf{x}') - E(\mathbf{x})E(\mathbf{x})' \)
- \( |V(\mathbf{x})| \neq 0 \) \( \Rightarrow \) no non-trivial riskless portfolio of risky assets.
- Thus, we rule out the existence of a safe asset \( x_0 = 1 \)
- Portfolio: \( p = \mathbf{w}'\mathbf{x} \)
- Portfolio weights: \( \mathbf{w} = (w_1, \ldots, w_N)' \)
- Expected value of a portfolio: \( E(p) = \mathbf{w}'E(\mathbf{x}) \)
- Variance of a portfolio: \( V(p) = \mathbf{w}'V(\mathbf{x})\mathbf{w} \)
- Cost of a portfolio: \( C(p) = \mathbf{w}'C(\mathbf{x}) \)
Portfolio selection

- Portfolio payoff space: $\mathcal{P} = \langle x \rangle$ (linear span of $x$).
- One important subset: $\mathcal{R} = \{ p \in \mathcal{P} : C(p) = 1 \}$, whose payoffs can be directly understood as returns per unit invested.
- If $C(x_i) \neq 0$, gross returns: $R_i = x_i / C(x_i)$, $C(R_i) = 1$
- Another important subset: $\mathcal{A} = \{ p \in \mathcal{P} : C(p) = 0 \}$, which is the set of all arbitrage (i.e. zero-cost) portfolios.
- Partition $x$ as $(x_1, x_{-1})$, where $x_{-1}$ is of dimension $n = N - 1$
- $\mathcal{A} = \langle r \rangle$, where $r = x_{-1} - R_1 C(x_{-1})$ is the vector of payoffs on the last $n$ risky assets in excess of the first one.
- In empirical work, excess returns are often computed by subtracting from gross returns a supposedly riskless asset, but the representation of $\mathcal{A}$ as the linear span of $r$ remains valid irrespective of the ordering of the original assets as long as $C(x_1) \neq 0$. 
Since $\mathcal{P}$ is a closed linear subspace of $L^2$, it is also a Hilbert space under the mean square inner product, $E(xy)$, and the associated mean square norm $\sqrt{E(x^2)}$, where $x, y \in L^2$.

Therefore, we can formally understand $C(.)$ and $E(.)$ as linear functionals that map the elements of $\mathcal{P}$ onto the real line.

The expected value functional is always continuous on $L^2$.

The cost functional is also continuous because of the law of one price.

Expected value representing portfolio:
$$E(p) = E(pp^o) \quad \forall p \in \mathcal{P} \Rightarrow p^o = E(x')[E(xx')]^{-1}x = P(1|\mathcal{P}) \text{ since } E(1 \cdot x) = E(x)$$

Cost representing portfolio:
$$E(p) = E(pp^*) \quad \forall p \in \mathcal{P} \Rightarrow p^* = C(x')[E(xx')]^{-1}x = P(m|\mathcal{P}) \text{ since } E(m \cdot x) = C(x)$$
Since we are assuming that no riskless asset exists, we can define an alternative expected value representing portfolio such that
\[ E(p) = Cov(p, \bar{\rho}^\circ) \quad \forall p \in \mathcal{P}. \]

Similarly, we can define an alternative cost representing portfolio such that \( C(p) = Cov(p, \bar{\rho}^*) \quad \forall p \in \mathcal{P}. \)

Not surprisingly,
\[
\begin{align*}
\bar{\rho}^* &= C(x')[V(x)]^{-1}x = p^* + [1 - E(p^\circ)]^{-1}C(p^\circ)p^\circ, \\
\bar{\rho}^\circ &= E(x')[V(x)]^{-1}x = [1 - E(p^\circ)]^{-1}p^\circ.
\end{align*}
\]
Unlike $\mathcal{R}$, $\mathcal{A}$ is a closed linear subspace of $\mathcal{P}$, and consequently a Hilbert space itself.

Given that $C(r) = 0$, the centred and uncentred cost representing portfolios $\delta^*$ and $a^*$ are both 0 in this case.

But we can still define the mean representing portfolios.

Specifically,

\[
\begin{align*}
\delta^\circ &= E(r')[E(rr')^{-1}]r, \\
a^\circ &= E(r')[V(r)]^{-1}r = [1 - E(a^\circ)]^{-1}a^\circ.
\end{align*}
\]
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Portfolio mean-variance frontiers

- \( \min_{\mathbf{w}} V(\mathbf{w}'\mathbf{x}) \) s.t. \( E(\mathbf{w}'\mathbf{x}) = \bar{\nu} \in \mathbb{R}, \ C(\mathbf{w}'\mathbf{x}) = \bar{\gamma} \in \mathbb{R}^+ \)

- For any \( p \in \mathcal{P} \), \( p = P(p|\langle \mathbf{p}^*, \mathbf{p}^0 \rangle) + u \)
  \( C(u) = E(up^*) = 0, \ E(u) = E(up^0) = 0 \)

- Hence, if we require \( E(p) = \bar{\nu} \) and \( C(p) = \bar{\gamma} \), then the solution is
  \[
  p^{MV}(\bar{\nu}, \bar{\gamma}) = P(p|\langle \mathbf{p}^*, \mathbf{p}^0 \rangle) \\
  = P(p|\langle \mathbf{p}^* \rangle) + P[p - P(p|\langle \mathbf{p}^* \rangle)|\langle \mathbf{p}^0 - P(p|\langle \mathbf{p}^* \rangle)\rangle] \\
  = \bar{\gamma} R^* + \rho(\bar{\nu}) a^0
  \]

  where \( R^* = p^*/C(p^*) \),

  \[
  a^0 = p^0 - P(p^0|\langle \mathbf{p}^* \rangle) = p^0 - [C(p^0)/C(p^*)] \cdot p^*
  \]

  \[
  \rho(\bar{\nu}) = \frac{\bar{\nu} - \bar{\gamma} E(R^*)}{E(a^0)}
  \]
Portfolio mean-variance frontiers

- Mean-variance frontier can be generated by $p^*$ and $p^\circ$ alone.
- Alternatively, we can generate it from $\beta^*$ and $\beta^\circ$:
  - Two-fund spanning
- Exceptions:
  1. $a^\circ = 0$ ("risk neutrality") The frontier collapses to $\tilde{\gamma} R^*$
  2. $\tilde{\gamma} = 0$ one-fund spanning with $a^\circ (C(a^\circ) = 0)$
- Usually, $\tilde{\gamma} = 1 \Rightarrow R_{MV}(\bar{\nu}) = p_{MV}(\bar{\nu}, 1)$
- This is the RMVF, which is a parabola in mean-variance space, and a hyperbola in mean-standard deviation space.
- But sometimes, $\tilde{\gamma} = 0 \Rightarrow r_{MV}(\bar{\mu}) = p_{MV}(\bar{\mu}, 0)$
- This is the AMVF, which is also a parabola in mean-variance space, but a straight line reflected at 0 in mean-standard deviation space.
min_{m \in L^2} V(m) \text{ s.t. } E(m) = \bar{c} \in \mathbb{R}^+, E(m \mathbf{x}) = C(\mathbf{x}).

If there were a riskless asset with gross return $1/\bar{c}$:

$m^{MV}(\bar{c}) = P(m|\mathcal{P}^+) = p^* + \alpha(\bar{c})(1 - p^\circ) = \alpha(\bar{c}) + \beta^* - \bar{c}\beta^\circ$

where $\mathcal{P}^+ = \langle 1, \mathbf{x} \rangle$ and

$$\alpha(c) = \frac{c - E(p^*)}{1 - E(p^\circ)} = c [1 + E(\beta^\circ)] - E(\beta^*),$$

If no riskless asset exists, then we simply repeat this calculation for all potential values of $c$, which implies that $m^{MV}(c)$ will be spanned by a constant, $p^* = P(m|\mathcal{P})$ and $(1 - p^\circ) = 1 - P(1|\mathcal{P})$ (or $\beta^*$ and $\beta^\circ$).

This frontier is also a parabola in mean - variance space for SDF's, and a hyperbola in mean - standard deviation space.
Duality

- Relationship with mean-variance analysis:
  \( m^{MV}(\bar{c}) - \alpha(\bar{c}) = p^* - \alpha(\bar{c}) p^\circ \) so \( m^{MV}(\bar{c}) \) can generally written as an affine transformation of \( R^{MV}(\bar{\nu}) \) for some \( \bar{\nu} \)

- There are two exceptions, which correspond to the asymptotes of the RMVF and SMVF.

- If we call \( R^{MV}_c \) the gross return associated to \( m^{MV}(c) \),
  \[
  \frac{|E(R) - 1/c|}{\sigma(R)} \leq \frac{|E(R^{MV}_c) - 1/c|}{\sigma(R^{MV}_c)} = \frac{\sigma[m^{MV}(c)]}{E[m^{MV}(c)]} \leq \frac{\sigma(m)}{E(m)}
  \]

- Absence of arbitrage opportunities: \( m^{MVA}(\bar{c}) = \max\{m^{MV}(\bar{c}), 0\} \)
In the case in which we only have access to arbitrage portfolios, then

\[ m^{MV}(c) = \frac{c}{1 - E(\alpha^\circ)}(1 - \alpha^\circ) = c\{1 - [\tilde{\alpha}^\circ - E(\tilde{\alpha}^\circ)]\}, \]

so in this case the frontier SDF’s are spanned by a single “fund”.

Still, this frontier will continue to be a parabola in mean - variance SDF space, but a straight line that starts from the origin in mean - standard deviation space.

The slope of this line is precisely the maximum (square) Sharpe ratio portfolio attainable over \( \mathcal{A} \).
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The effects on the SMVF and RMVF of including additional assets

- $x_1$: original assets ($N_1 \times 1$)
- $x_2$: new assets ($N_2 \times 1$)
- $x = (x'_1 \ x'_2)'$: expanded set of assets ($N \times 1$, $N = N_1 + N_2$)
- RMVF will shift to the left because our investment opportunities improve.
- SMVF will rise because there is more information in the data about $m$.
- However, this is not always the case: $x_1$ spans the RMVF and/or SMVF generated from $x$ when the old and new frontiers are the same.
- A third and last possibility, which we call tangency, is that the frontiers share a single point.
- In the case of arbitrage portfolios, tangency is equivalent to spanning.
Frontiers with a countably infinite number of primitive assets

- $\overline{P}$: closure of the set of payoffs from all possible portfolios of the primitive assets $\bigcup_{N=1}^{\infty} P_N$

- Problems: Even if $E(xx')$ has full rank for all $N$, there may exist:
  1. Limiting riskless unit-cost portfolios of risky assets
  2. Limiting arbitrage opportunities

- Once we rule out those possibilities, our previous analysis in terms of representing portfolios applies.