Bilateral Mechanism Design: Practical Contracting in Multi-Agency

Yu Chen
Nanjing University

June 2, 2016
Bilateral Mechanism Design: Practical Contracting in Multi-Agency*

Yu Chen†

Abstract

We modelize and investigate the analytical rationale of employing bilateral mechanism design, which simplifies collective mechanism design by ignoring relative information evaluation, in generalized multi-agency contracting games under Bayesian Nash equilibrium. We permit interdependent valuations, contract externalities, correlated types, and heterogeneous or different message sets of different agents. The delegation principle under Bayesian Nash equilibrium identifies that bilateral Bayesian mechanism design can be translated to delegated Bayesian menu design without loss of generality. We take advantage of interim-payoff-equivalence to provide economically interesting conditions on the primitives for the full equivalence in which bilateral mechanism design can be substituted for collective mechanism design. Our analysis can also incorporate individual rationality constraints. Moreover, we discuss the approximation of full equivalence and the case allowing primitive constraints across the contracts for different agents.

Keywords: Bayesian Nash equilibrium, mechanism design, menu design, delegation principle, interim-payoff-equivalence

JEL Classification: C72 D82 D86

---

*This paper is a substantial revision of Chapter II of my Ph.D. dissertation. I would like to thank Frank Page and Robert Becker for advice, encouragement and comments. For additional helpful remarks, suggestion and comments, thanks are also due to Alessandro Pavan, Johannes Horner, Haomiao Yu, Yongchao Zhang, Rongzhu Ke, Xiang Sun, Geoffrey Woglom, Jun Zhang, Nian Yang, Jie Zheng, Zhiqi Chen, Mingjun Xiao, Masaki Aoyagi, Aimin Huang, and the participants/audiences at University of Queensland, Tsinghua University, Sun Yat-sen University, Shanghai University of Finance and Economics, Osaka University, APET annual meeting 2014 and SAET annual conference 2014. However I am solely responsible for any errors.

†School of Economics, Nanjing University, 16 Jinyin Street, Nanjing, Jiangsu, China 210093. Email: yourchenyu@gmail.com.
1 Introduction

Bilateral mechanisms, as a simplified class of collective mechanisms by ignoring relative information evaluation, have attracted much attention in the literature of contract and mechanism design over recent years. The canonical mechanisms in typical analysis of mechanism design are collective mechanisms, which specifies each contract (or allocation) for an individual agent with the joint reports of all agents. In contrast, bilateral mechanisms specify each contract for an individual agent merely with his individual report.

A few facts or arguments hint that bilateral mechanism design would be a more practical solution to deal with private, decentralized information due to its simplification. The classic theory of mechanism design shows how one central planner can theoretically succeed in designing revelation mechanisms subject to incentive constraints to allocate resources in a centralized way to deal with the economic agents with private, decentralized information (knowledge). In practice, however, Friedrich August von Hayek’s warning about overconfidence of centralization in his influential paper (1945) is still relevant. Designing and implementing real world revelation mechanisms seem unreasonably complicated, due to informational complexity, limited information processing capacities of human beings, etc. Consistent with Hayek’s concern, McAfee and Schwartz (1994) also point out that designing a complete and comprehensive multilateral (collective) contract or mechanism might be indeed practically demanding, and the associated costs of auditing and processing information might significantly rise with the number of parties involved. In their recent paper, Dequiedt and Martimort (2015) further remark that in addition to saving on haggling and transaction costs bilateral contracts are also the only feasible arrangements when antitrust laws preclude multilateral agreements.

Although many authors directly adopt bilateral mechanisms as the analytical object in their contracting contexts, few of them formulate or modelize what facilities bilateral mechanisms may indeed bring relative to canonical collective mechanisms, given the fact that collective mechanism design can make the principal at least as well off as bilateral mechanism design. It is unclear why a principal wants to rely on bilateral mechanisms when collective mechanisms are available. What facilities can the simplified form of bilateral mechanisms specifically provide for contracting procedure relative to collective mechanisms either from the perspective of analytical simplification or from the perspective of actual effect? This paper is aimed to use explicit modelized analysis to answer this research question.

This paper modelizes and investigates the analytical rationale of employing bilateral mechanisms in generalized multi-agency contracting games under Bayesian Nash equilibrium. Multi-agency denotes that a single principal contracts with multiple agents. It is frequently observed in many real-life cases. The multi-agency contracting games in our consideration are essentially two-stage pure-strategy games. Since contracting games with pure strategies are more realistic

---

1 See McAfee and Schwartz (1994), Segal (1999), Han (2006), Hansen and Motta (2012), Dequiedt and Martimort (2015) among many others.
in practice, we focus on them in this paper. In mechanism design procedure, the principal’s strategy is to offer mechanisms, which are mappings from the agents’ reports to contracts for the agents, to the agents. Each pre-offered mechanism defines a non-cooperative subgame for all agents to play simultaneously. Bayesian Nash equilibrium is the solution concept used for such a subgame. The principal seeks an optimal mechanism from a class of mechanisms inducing a subgame for the agents in which some particular reporting profile of the agents is achieved as Bayesian Nash equilibrium. Such class of mechanisms are called Bayesian mechanisms. ²

We consider a multi-agency environment in great generality of the pure adverse selection model.³ Unlike single-agency, multi-agency suggests significant interaction and interdependence between the agents. The agents’ types can be correlated. Contract sets for individual agents can be heterogeneous. Each agent’s (expected) payoff can depend not only on his own type and specified contract but also on those of the other agents. The first case is called information externalities (or interdependent valuations), and the second case is called contract externalities. These two points have attracted attention in a number of recent studies.⁴ Such features imply "full-blown interdependence" among the agents. The message sets of different agents can be heterogeneous or different. Moreover, the principal’s (expected) payoff is permitted to depend jointly on all the agents’ types and specified contracts. Hence, the agents will behave strategically, and the impacts of their respective asymmetric information will be interrelated.

This paper establishes two major findings to explain the facilities of adopting bilateral mechanisms in multi-agency contracting. First, under Bayesian Nash equilibrium bilateral mechanism design can always be translated to delegated menu design without loss of generality even in the general multi-agency contracting environment. In delegated menu design, the principal can simplify the information communication for specifying contracts to the agents. She simply offers (joint) menus, which are sets of contract profiles of the agents. The agents are entitled to directly select the contracts for themselves within the pre-offered menu. Each pre-offered menu also defines a non-cooperative subgame for all agents to play simultaneously with Bayesian Nash equilibrium as its solution concept. Such a menu is call a Bayesian menu. The principal seeks an optimal Bayesian menu, in which different agents can be allowed to separately choose the contracts within the individual-specified menus on their own accord. Menu design is an even "simpler" procedure in practice. Menu design procedure actually suggests delegating to the agents the decision rights to specify contracts and simplifying information communication required for the central designer to specify the contracts. Our delegation principle identifies the

²This paper focuses on the mechanism design under Bayesian Nash equilibrium, while Chen and Wu (2015) study the delegation principle under ex post equilibrium. The importance of Bayesian mechanism design results from two reasons. First, the parties in a real world contracting game may still have finer information with respect to Bayesian updating. This will indeed provide more leeway for full equivalence compared with ex post (or dominant-strategy) equilibrium. Second, (nontrivial) Bayesian mechanism is more likely to exist under the generalized models than ex post (or dominant-strategy) mechanism.

³It is not technically demanding to include moral hazard in this context. One can add one more part in the mechanism as the action recommendation mechanism specifying action recommendations to the agents for any given their reports. See Kadan et al. (2014).

bilateral Bayesian mechanism design is strategically equivalent to the Bayesian menu design. Specifically, equilibrium payoffs for PL and agents under the game induced by each Bayesian menu will remain the same as the game induced by some bilateral mechanism, and vice versa. Furthermore, optimal bilateral Bayesian mechanism design is equivalent to optimal Bayesian menu design.

Second, our analysis provides economically interesting conditions for the full equivalence between (optimal) bilateral and collective Bayesian mechanism designs. It is still likely that the optimal bilateral Bayesian mechanism (and optimal Bayesian menu) can make the principal as well off as the optimal collective Bayesian mechanism, although the latter will clearly yield to the principal an optimal (expected) payoff at least as large as the former does. In that case, we can use bilateral mechanism design or menu design to substitute collective mechanism design with no loss of generality. Information structure with respect to Bayesian updated beliefs provides a possibility for the full equivalence when the agents have no contract externalities and the principal can separately draw welfare from contracts taken by different agents. In this vein, we find that the collection of bilateral mechanisms interim-payoff-equivalent to all collective BIC mechanisms is exactly the collection of all bilateral BIC mechanisms in the quasi-separable environment. Then when the principal’s payoff is related with the agents’ payoffs in a linear or some particular nonlinear form, the full equivalence between optimal bilateral and collective Bayesian mechanism designs can be established.

We also address several extensions based on our findings. (1) It is not technically difficult to incorporate the individual rationality conditions in our results. (2) We show that the value of our findings may lie in approximation of full equivalence, i.e., approximating collective BIC mechanism design by bilateral BIC mechanism design or Bayesian menu design, even if the full equivalence does not exactly hold. (3) We discuss the full equivalence under explicit primitive constraints across the contracts for different agents.

Interim-payoff-equivalence is an important concept and analytical tool in the Bayesian mechanism design literature. Recently, Manelli and Vincent (2010), and Gershkov et al. (2013) apply it to examine the equivalence of Bayesian and dominant strategy mechanism design. In contrast, this paper takes a new direction of the application of interim-payoff-equivalence. The idea behind the equivalence results of this paper is similar to theirs: a posteriori simplification of multi-agency contracting procedure can be offset by finer a priori information (common knowledge) structure in terms of Bayesian updated beliefs and certain specific contracting environment.

Han (2006) also addresses the equivalence of mechanism and menu designs under Perfect Bayesian Equilibria in a mixed-strategy multi-principal multi-agent "bilateral" environment with private valuations and homogeneous payoff forms of the agents. All "bilateral" mechanisms the principals can offer to each agent are restricted to be the functions from a single uniform message

---

5Quasi-separable environment is an extension of the separable environment introduced in Chung and Ely (2006) and is useful in many economic applications and previous studies.
set across the agents to an independent set of randomized contracts only available to that agent. Each agent’s reporting strategy is also randomized. This paper extends Han’s model to a more general setting in multi-agency games\(^6\) and focus on pure-strategy specification, which is more frequently observed in most applications. We also takes into account the related scenarios when the mechanisms are allowed to be "collective."

Moreover, Dequiedt and Martimort (2015) extend Han’s setting to allow message sets of different agents to be heterogeneous or different subsets of Euclidean spaces and examine how the principal acts opportunistically in a single-principal-two-agent bilateral pure-strategy vertical contracting environment with private valuations and Bayesian Nash Equilibrium. But all agents' types belong to an identical unidimensional real closed interval. In addition, they automatically equate bilateral mechanisms with menus in the context of bilateral mechanisms. Their study is not aimed at a substantive examination on why bilateral mechanisms can be equivalent to menus or why a principal will stick to bilateral mechanisms when collective mechanisms are still possible to consider in a generalized multi-agency environment. Our work can be regarded as an important complement or extension of their study.

The rest of the paper is organized as follows. Section 2 presents the primitives of our model. Section 3 formally formulates the bilateral and collective Bayesian incentive compatible mechanism designs. A simple motivating example similar to Dequiedt and Martimort (2015) is presented in Section 4. Section 5 shows the equivalence of bilateral mechanisms and menus in the contracting games under Bayesian Nash Equilibrium. In Section 6, we thoroughly examine the full equivalence between bilateral and collective Bayesian incentive compatible mechanism designs. Some discussions are given in Section 7. Conclusion is given in section 8. All proofs of the propositions and lemmas are relegated to the appendix.

## 2 Primitives

We consider a pure-strategy multi-agency contracting game with one principal (short for PL, female) and \(n\) agents indexed by \(i \in \mathcal{N} = \{1, \cdots, n\}\).\(^7\) PL moves first. Then the agents follow simultaneously and behave non-cooperatively.

Agent \(i\) (short for \(A_i\), male) has some private payoff type \(\theta_i \in \Theta_i\), where \(\Theta_i\) is a complete separable metric space, i.e., Polish space.\(^8\) We write \(\theta = (\theta_i)_{i \in \mathcal{N}} \in \Theta = \prod_{i=1}^{n} \Theta_i\) and \(\theta_{-i} = (\theta_j)_{j \in \mathcal{N} \setminus \{i\}} \in \Theta_{-i} = \prod_{j \neq i} \Theta_j\).\(^9\) Let \(\mu_i\) be a probability measure defined on \(\Theta_i\) and \(\mu\) be a probability measure on \(\Theta\). \(\mu\) characterizes the common prior over the agents’ types, which are allowed to be

---

\(^6\)In contrast, we allow the "full-blown interdependence" on the agents, and mechanisms with heterogeneous message sets of different agents. Those situations have been broadly studied in mechanism design literature. Yet Han’s results cannot directly apply there.

\(^7\)The model setup and major definitions in this paper are similar to Chen and Wu (2016).

\(^8\)A familiar example of Polish space is any Euclidean space, \(\mathbb{R}^n\). Note that any open or closed subset of a Polish space is still Polish. Any compact metric space is also a Polish space.

\(^9\)Note that \(\Theta\) and \(\Theta_{-i}\) are also Polish spaces.
correlated. Let \( \mu_{-i}(\cdot | \theta_i) \) denote a conditional probability measure on \( \theta_i \) over \( \Theta_{-i} \) and represent \( A_i \)'s interim (Bayesian updated) belief about the other players' types after learning her own type \( \theta_i \).\(^{10}\) If the type spaces are subsets of Euclidean spaces, all the relevant probability measures can be equivalently represented by the corresponding probability distributions. For each \( i \) and each closed subset \( A \) of \( \Theta_{-i} \), \( \mu_{-i}(A \cdot) \) is continuous on \( \Theta_i \).\(^{11}\)

The contract\(^{12}\) available to \( A_i \) is \( k_i \in K_i.\(^{13}\) The set of all possible joint contracts is given by the (joint) contract set \( K = \prod_{i=1}^{n} K_i \).\(^{14}\) Its typical element is \( k = (k_i)_{i \in \mathcal{N}} \). Write \( k_{-i} = (k_j)_{j \in \mathcal{N} \setminus \{i\}} \). The set \( K \) is assumed to be a Polish space. Several typical examples fitting these assumptions on (joint) contract sets are shown below.\(^{15}\)

**Example 1 (Finite contract sets)** There are only finitely many contracts in each \( K_i \). \( K = \prod_{i=1}^{n} K_i \) must be finite and therefore a compact metric space and therefore a Polish space.

**Example 2 (Product-price pairs in nonlinear pricing)** Each buyer \( i \) is offered a product-price pair \((x_i, p_i)\). \( x_i \) is some product characteristics, such as quantity, quality, etc. \( p_i \) is the price the seller can charge for \( i \). So the joint contract set can be

\[
K = \{(x_1, \ldots, x_n, p_1, \ldots, p_n) \in \mathbb{R}^n \times \mathbb{R}^n : x_i \geq 0, p_i \geq 0\}.
\]

\( K \) is clearly a Polish space.

**Example 3 (State-contingent contract sets)** The state is \( \omega \in \Omega \subseteq \mathbb{R} \). \( \eta \) is a probability measure over \( \Omega \). Assume all the contracts are outcome-contingent. If for each \( i \), (1) \( K_i \) is a subset of the collection of all Borel-measurable functions from \( \Omega \) to \([L,H] \subseteq \mathbb{R}\) indexed by a compact metric space \( I \), that is, \( K_i = \{f(\cdot, \gamma) : \Omega \rightarrow [L,H] | \gamma \in I, \text{ and } f \text{ is Borel measurable in } \omega \text{ and continuous in } \gamma\} \), where \( I \) is a compact metric space, and (2) \( K_i \) contains no redundant contracts, that is, if for any two \( k_i \) and \( k^{\prime}_i \) in \( K_i \) satisfying \( k_i(\omega') \neq k^{\prime}_i(\omega') \) for some \( \omega' \in \Omega \), \( \eta(\{\omega \in \Omega : k_i(\omega) \neq k^{\prime}_i(\omega)\}) > 0 \), then \( K_i \) is a compact metric space by Proposition 1 in Tulcea (1973). So is \( K \).

Let \( v_i : K \times \Theta \rightarrow \mathbb{R} \) denote \( A_i \)'s payoff function defined over contract profiles and type profiles. \( v_i \) is continuous on \( K \) and Borel-measurable on \( \Theta \). \( A_i \)'s interim payoff function \( V_i : K \times \Theta_i \rightarrow \mathbb{R} \)

---

\(^{10}\)Note that \( \mu_{-i}(\cdot | \theta_i) \) is not necessarily derived from the prior \( \mu \) on \( \Theta \). Moreover, we can even allow heterogeneous beliefs/priors across all parties.

\(^{11}\)One typical example for this assumption is that \( \mu_{-i} \) has the conditional density \( f(\theta_{-i}|\cdot) \) which is continuous over \( \theta_i \).

\(^{12}\)Some authors may also call it outcome, alternative, or allocation. But we use the term "contract" here to literally take on richer meanings, for instance, outcome (state)-contingent contracts can also be considered as a kind of contracts, and to be consistent with the term "contracting" games or procedures.

\(^{13}\)\( K_i \) may contain an element \( k_0 \) which denotes "no contracting."

\(^{14}\)In an even more general setting, we can allow \( K \) to be any subset of \( \prod_{i=1}^{n} K_i \). In other words, we allow the explicit primitive constraints across the contracts for different agents. This will be discussed in Section 7.

\(^{15}\)They are also provided in Chen and Wu (2015).
is defined by

\[ V_i(k, \theta_i) = \int_{\Theta_{-i}} v_i(k, \theta) \mu_{-i}(d\theta_{-i} | \theta_i). \]

Let \( u : K \times \Theta \to \mathbb{R} \) denote PL’s payoff function over contract and type profiles. \( u \) is continuous on \( K \) and Borel-measurable on \( \Theta \).

### 3 Bayesian Mechanism Design

The canonical contracting procedure to deal with adverse selection in multi-agency is the centralized mechanism design. Accordingly, the principal-agent contracting game over mechanisms is played as follows:

In stage 1, PL proposes to the agents a commonly observable mechanism.

In stage 2, the agents unilaterally learn their own true types and simultaneously send reports to PL.

In stage 3, through the pre-offered mechanism, PL assigns contracts to the agents after learning their reports.

In stage 4, after the agents’ participation\(^{[16]}\), the contracts are simultaneously executed.

Due to legal customs, technological restriction, or other reasonable constraints, either collective mechanisms or bilateral mechanisms may be available \(a priori\) to PL. They are defined below.

**Definition 1** A **collective mechanism** is a list of Borel measurable functions \( k = (k_i : \Theta \to K_i)_{i \in N} \) satisfying for each \( \theta \in \Theta \), \((k_1(\theta), \cdots, k_n(\theta)) \in K\), where each of its component \( k_i \) specifies a contract to \( A_i \) for each type report profile of all agents. A **bilateral mechanism** is a list of Borel measurable functions \( \mathbf{k} = (k_i : \Theta_i \to K_i)_{i \in N} \) satisfying \((k_1(\theta_1), \cdots, k_n(\theta_n)) \in K\), where each of its component \( k_i \) specifies a contract to \( A_i \) for each type report profile of single \( A_i \). Let \( \mathcal{F}(\Theta, K) \) and \( \mathcal{F}(\Theta, K) \) respectively denote the collection of collective mechanisms and that of bilateral mechanisms.

**Remark 1** The well-known revelation principle allows us to restrict attention to Bayesian incentive compatible direct mechanisms out of general Bayesian mechanisms\(^{[17]}\). Thus, we focus on direct mechanisms in this paper.

Collective mechanisms evaluate relative information (all agents’ type reports) for specifying every individual agent’s contracts in nature, whereas bilateral mechanisms ignore it and merely evaluate absolute information (every individual agent’s type reports) for specifying every individual agent’s contracts.

Each mechanism offered by PL induces a simultaneous-moved subgame for the agents in which **Bayesian Nash equilibrium (BNE)** is considered as the solution concept.

\(^{[16]}\) It is permitted that not all the agents eventually participate.

\(^{[17]}\) Easy to check the revelation principle also holds for bilateral mechanisms.
Definition 2 A collective mechanism $\mathbf{k}$ is **Bayesian incentive compatible (BIC)** if it induces truthful reporting as the BNE for all the agents, i.e., for each $i \in \mathcal{N}$, $\theta_i \in \Theta_i$, $\theta'_i \in \Theta_i$,

$$
\int_{\Theta_{-i}} v_i(\mathbf{k}(\theta), \theta) \mu_{-i}(d\theta_{-i}|\theta_i) \geq \int_{\Theta_{-i}} v_i(\mathbf{k}(\theta'_i, \theta_{-i}), \theta) \mu_{-i}(d\theta_{-i}|\theta_i).
$$

A bilateral mechanism $\mathbf{\bar{k}}$ is **BIC** if it induces truthful reporting as the BNE for all the agents, i.e., for each $i \in \mathcal{N}$, $\theta_i \in \Theta_i$, $\theta'_i \in \Theta_i$,

$$
\int_{\Theta_{-i}} v_i(\mathbf{\bar{k}}(\theta), \theta) \mu_{-i}(d\theta_{-i}|\theta_i) \geq \int_{\Theta_{-i}} v_i(\mathbf{\bar{k}}(\theta'_i, \theta_{-i}), \theta) \mu_{-i}(d\theta_{-i}|\theta_i).
$$

Thus, two corresponding PL’s optimization problems address contracting games over Bayesian mechanisms below:

The optimal collective BIC mechanism design problem is (**P1**)

$$
\max_{\mathbf{k} \in \mathcal{F}(\Theta, \mathcal{K})} \int_{\Theta} u(\mathbf{k}(\theta), \theta) \mu(d\theta)
$$

s.t. $\mathbf{k}$ is BIC.

The optimal bilateral BIC mechanism design problem is (**P2**)

$$
\max_{\mathbf{\bar{k}} \in \mathcal{F}(\Theta, \mathcal{K})} \int_{\Theta} u(\mathbf{\bar{k}}(\theta), \theta) \mu(d\theta)
$$

s.t. $\mathbf{\bar{k}}$ is BIC.

4 A Motivating Example

We consider a simple vertical contracting example similar to Dequiedt and Martimort (2015) for the motivation of our subsequent analysis. A upstream manufacturer contracts with $n$ downstream retailers indexed by $i = 1, \cdots, n$. Each retailer $i$ can observe a downstream market valuation signal $\theta_i \in \Theta_i \subseteq \mathbb{R}$, which is the private information of $i$. The manufacturer sells to retailer $i$ a quantity of a good $x_i \in X_i$, where $X_i$ is a closed subset of $\mathbb{R}$, at price $t_i \in T_i$, where $T_i$ is a closed subset of $\mathbb{R}$. The manufacturer has a payoff function

$$
u(x, t, \theta) = \sum_{i=1}^{2} t_i - C(x),$$

where $C : X \rightarrow \mathbb{R}$ is her cost function. Retailer $i$ has a payoff function

$$v_i(x, t, \theta) = \pi_i(x, \theta) - t_i,$$
where \( \pi_i : X \times \Theta \to \mathbb{R} \) is his profit function on his downstream market. A collective mechanism is a list of Borel-measurable functions \((x, t) = ((x_i : \Theta \to X_i)_{i \in \mathcal{N}}, (t_i : \Theta \to T_i)_{i \in \mathcal{N}})\). A bilateral mechanism is a list of Borel-measurable functions \((\bar{x}, \bar{t}) = ((\bar{x}_i : \Theta_i \to X_i)_{i \in \mathcal{N}}, (\bar{t}_i : \Theta_i \to T_i)_{i \in \mathcal{N}})\).

In this case, it is usually information demanding to adopt collective mechanisms in real life, especially as the number of retailers increases. In contrast, bilateral mechanisms are the simpler contracting mechanisms in the functional forms. Dequiedt and Martimort (2015) presumably consider using bilateral mechanisms to analyze the information opportunism problem. As they point out, "in the standard formalism (of the vertical contracting literature), the vertical organization is run through a set of bilateral contracts (mechanisms)." Nevertheless, it is unclear what facilities the simple form of bilateral mechanisms can specifically bring in contracting practices either from the analytical perspective or from the perspective of actual effects.

Two major points are not clearly discussed in their analysis. First, when the authors employ bilateral mechanisms, they automatically equate bilateral mechanisms with menus, which are usually regarded as sets of contracts in literature. Yet the validity of such equivalence is unclear in the general settings. Second, unless there is some ad hoc realistic or analytical restrictions, it is still reasonable to adopt collective mechanisms in principle. Therefore, it is desirable to know when bilateral mechanism design can be equivalent to traditional collective mechanism design, that is, when we can use bilateral mechanisms to substitute collective mechanisms without loss of generality. These two points motivate our subsequent analysis on the relevant equivalence results, and these will be connected with the practical facilities of adopting bilateral mechanisms.

5 Equivalence of Bilateral Mechanisms and Menus

The first practical facility of adopting bilateral mechanism design in multi-agency is that we can always equivalently translate bilateral mechanism design to menu design, which is a simple, delegated contracting procedure in practice. In this sense, bilateral mechanisms can be regarded as the mathematical representation of menus in the form of mechanisms.

5.1 Bayesian Menu Design

Let us formulate the delegated menu design procedure formally. In menu design, PL avoids to process decentralized information or have the agents send messages to specify contracts for the agents. Instead she can design for the agents a (joint) menu, i.e., a subset of the joint contract set, and allow them to simultaneously pick the contracts from the menu.

Accordingly, the principal-agent contracting game over mechanisms unfolds as follows:

At stage 1, PL proposes to the agents a joint menu, which is commonly observable.

---

18 We will formally define and discuss the menu design procedure in section 5.
19 Some authors may also use the term "catalogs" instead.
At stage 2, the agents unilaterally learn their own true types and simultaneously select the contracts from the pre-offered joint menu.

At stage 3, after the agents’ participation, the contracts are simultaneously executed.

Each individual-specific contract menu \( C_i \) is a nonempty, closed subset of \( K_i \). A joint contract menu \( C \), as a product of individual menus, is a subset of \( K \). The set of all joint contract menus is

\[
P_f(K) = \{ C = (C_1, \cdots, C_n) \subseteq K | C_i \text{ is a nonempty, closed subset of } K_i \}.
\]

Each \( A_i \)'s strategy is a Borel measurable function \( \tilde{k}_i : \Theta_i \rightarrow K_i \) which denotes \( A_i \)'s contract selection according to his type. Let \( F_i \) denote the collection of all possible \( \tilde{k}_i \)'s. Write \( \tilde{k} = (\tilde{k}_i)_{i \in \mathcal{N}}, \tilde{k}(\theta) = (\tilde{k}_i(\theta_i))_{i \in \mathcal{N}}, \text{ and } \tilde{k}_{-i}(\theta_{-i}) = (\tilde{k}_j(\theta_j))_{j \in \mathcal{N}\setminus \{i\}}. \)

The contract selection profile under a menu \( C \in P_f(K) \) is \( \tilde{k} \in F_c \), where

\[
F_c = \{ \tilde{k} \in \prod_{i=1}^{n} F_i | \tilde{k}(\theta) \in C \text{ for each } \theta \in \Theta \}.
\]

Each agent is entitled to select a contract from his individual-specific menu \( C_i \) offered by PL. Each joint menu \( C \) offered by PL induces a simultaneous-moved subgame played by the agents with BNE as the solution concept. All agents observe all the possible optional contract profiles in each preoffered menu. A certain contract profile within the menu needs to be simultaneously agreed on by the agents as the BNE.

**Definition 3** A contract selection profile \( \tilde{k} \in F_c \) is a BNE under a joint menu \( C \) if for each \( i \in \mathcal{N}, \theta_i \in \Theta_i \),

\[
\int_{\Theta_{-i}} v_i(\tilde{k}(\theta), \theta)\mu_{-i}(d\theta_{-i}|\theta_i) \geq \int_{\Theta_{-i}} v_i(\tilde{k}'_i(\theta_i), \tilde{k}_{-i}(\theta_{-i}), \theta)\mu_{-i}(d\theta_{-i}|\theta_i),
\]

for all \( \tilde{k}'_i \in F_i \) satisfying \( \tilde{k}'_i(\theta_i) \in C_i \). Such a joint menu \( C \) is called a Bayesian (joint) menu.

Thus, PL can simply design individual-specified (product) menus and permit different agents to separately choose the contracts within the individual-specified menus on their own accord. PL can deduce that the agents will have the BNE contract selection profile in the subgame defined by a Bayesian menu and hence faces an optimization problem to address this contracting game, that is, optimal Bayesian menu design problem (P3):

\[
\max_{C \in P_f(K)} \int_{\Theta} \max_{k \in F_c} u(\tilde{k}(\theta), \theta)\mu(d\theta)
\]
\[
s.t. \ k \text{ is the BNE under } C.
\]

In view of tie-breaking, PL may designate or recommend \( \tilde{k} \) in her best interest for the agents with type profile \( \theta \) to follow.
5.2 Delegation Principle under Bayesian Nash Equilibrium

An important observation summarized in the delegation principle under Bayesian Nash equilibrium is a complete characterization of all bilateral BIC mechanisms via Bayesian menus even in a general multi-agency situation permitting "full-blown" interdependence, including correlated types, externalities in contracts, and interdependent valuations.

First consider the set-valued mapping $\Psi : \Theta \times P_f(\mathcal{K}) \to \mathcal{K}$ defined by

$$\Psi(\theta, C) = \{ k \in C : k = \tilde{k}(\theta) \text{ such that } \tilde{k} \text{ is a BNE under } C \}.$$  

It is deducible by PL and represents the $\theta$-type-profile agents’ joint BNE response to any menu offer $C$. A well-defined $\Psi(\cdot, C)$ implies that there exists at least one Bayesian menu $C$.

**Definition 4** Given $C \in P_f(\mathcal{K})$, $\Psi(\cdot, C)$ is **well-defined** if $\Psi(\cdot, C)$ is nonempty for each $\theta$.

We also need to introduce two assumptions below.

**Assumption 1** For each $i \in \mathcal{N}$, $v_i$ is continuous in $\theta_{-i} \in \Theta_{-i}$, and $v_i(k, \theta_i, \cdot)$ is bounded for given $k$ and $\theta_i$ on $\Theta_{-i}$.

**Assumption 2** $\mathcal{K}$ is a compact metric space.

**Proposition 1 (Delegation Principle under Bayesian Nash Equilibrium)** Under Assumption 1, given a contracting mechanism $\tilde{k} \in \mathcal{F}(\Theta, \mathcal{K})$,

(i) if $\tilde{k}$ is BIC, then $C = \prod_{i=1}^{n} \text{cl}\{(\tilde{k}_i(\theta_i) : \theta_i \in \Theta_i \}$ is such that $\Psi(\cdot, C)$ is well-defined, and $\tilde{k}$ is a Borel-measurable selection from $\Psi(\cdot, C)$, that is, $\tilde{k}(\theta) \in \Psi(\theta, C)$ for all $\theta \in \Theta$, and

(ii) if there exists a joint menu $C \in P_f(\mathcal{K})$ such that $\Psi(\cdot, C)$ is well-defined, and $\tilde{k}$ is a Borel-measurable selection from $\Psi(\cdot, C)$, then $\tilde{k}$ is BIC.

This delegation principle identifies that whenever we consider a subgame induced by a bilateral BIC mechanism $\tilde{k}$ we can equivalently consider a subgame induced by a Bayesian menu $C = \prod_{i=1}^{n} \text{cl}\{(\tilde{k}_i(\theta_i) : \theta_i \in \Theta_i \}$. The converse is also true. Type-profile-$\theta$ agents will be induced to choose the same contract profiles ex post under both contracting procedures. More specifically, equilibrium payoffs for PL and agents under the game induced by each Bayesian menu will remain the same as the game induced by some bilateral BIC mechanism, and vice versa. So designing bilateral BIC mechanisms is equivalent to designing Bayesian menus, which simplifies contracting procedure by entitling the agents to directly select the contracts in practice.

Based on Proposition 1, Proposition 2 further indicates that optimal Bayesian menu design is strategically equivalent to optimal bilateral Bayesian mechanism design.

**Proposition 2** Under Assumption 1 and 2, we have

(i) if $\tilde{k}^*$ solves the contracting problem over bilateral BIC mechanisms given by $(P2)$, then $C^* = \prod_{i=1}^{n} \text{cl}^{20}\{(\tilde{k}_i^*(\theta_i) : \theta_i \in \Theta_i \}$ solves the contracting problem over Bayesian menus given by $P_f(\mathcal{K})$. 

\[^{20}\text{cl} \text{ denotes the closure.}\]
(P3), and

(ii) If $C^*$ solves (P3), then $\overline{k}^*$ satisfying $\overline{k}^*(\theta) \in \arg\max_{\overline{k}(\theta) \in \Psi(\theta, C^*)} u(\overline{k}(\theta), \theta)$ for each $\theta \in \Theta$ solves (P2).

Moreover, the optimal objective values of the two problems are equal.

Recall the motivating example. An individual menu for retailer $i$ is a nonempty, closed subset $C_i \subseteq X_i \times T_i$. By our analysis, designing bilateral BIC mechanisms is actually equivalent to designing Bayesian menus now. In this respect, vertical contracting can avoid the process of complex, indirect information reports and be straightforward directed to the equilibrium quantity-transfer pair(s) according to the agents’ separate choices. This also justifies the presumed equivalence of bilateral mechanisms and menus in Dequiedt and Martimort (2015).

6 Full Equivalence of Bilateral Mechanisms and Collective Mechanisms

Obviously, the collection of bilateral BIC mechanisms is essentially equivalent to a subset of the collection of collective BIC mechanisms. Therefore, the optimal collective mechanism will clearly make PL at least as good as the optimal bilateral mechanism does. But this is just a weak dominance result. It does not completely rule out the possibility of the full equivalence of bilateral mechanisms and collective mechanisms. If bilateral mechanism design is equivalent to collective mechanism design, especially when the optimal bilateral mechanism brings to PL the same objective value (expected payoff) as the optimal collective mechanism does, then PL will have incentive to adopt bilateral mechanisms to be a practical contracting procedure relative to adopting collective mechanisms. Hence, our subsequent discussion on full equivalence boils down to explore the conditions under which bilateral BIC mechanism design is equivalent to the collective BIC mechanism design.

6.1 Interim-Payoff-Equivalent Mechanisms

It is desirable to find certain economically intuitive conditions on the primitives for this full equivalence. A key idea is that such simplification of the contracting procedure tends to be offset by finer information structure (common knowledge) in the contracting environment. In light of this idea, we can identify a class of economically intuitive conditions in which the full equivalence holds by introducing interim payoff equivalence in Bayesian implementation.

**Definition 5** A collective mechanism $k \in F(\Theta, K)$ and a bilateral mechanism $\overline{k} \in \overline{F}(\Theta, K)$ are interim-payoff-equivalent (IPE) with respect to $A_i$ if for each $\theta_i \in \Theta_i$,

$$\int_{\Theta_{-i}} v_i(\overline{k}(\theta), \theta) \mu_{-i}(d\theta_{-i}|\theta_i) = \int_{\Theta_{-i}} v_i(k(\theta), \theta) \mu_{-i}(d\theta_{-i}|\theta_i).$$
By proof of construction, Lemma 1 shows that for each collective mechanism its IPE bilateral mechanism may exist under the Bayesian (interim) belief structure.

**Lemma 1** If for each \( i \in \mathcal{N} \), suppose

(i) \( K_i \) is a connected, locally compact Polish space,\(^{21}\) and

(ii) \( v_i(k, \theta) \equiv v_i(k_i, \theta) \),

then for each collective (respectively, collective BIC) mechanism \( k \), there exists a bilateral (respectively, bilateral BIC) mechanism \( k \) IPE (with respect to all agents) to \( k \) if and only if \( v_i \) satisfies **boundary preserving property** under integration over \( \mu_{-i} \) and \( K_i \), that is, for each \( \theta_i \),

\[
\begin{align*}
\sup_{k_i \in K_i} \int_{\Theta_{-i}} v_i(k_i, \theta) \mu_{-i}(d\theta_{-i}|\theta_i) &= \int_{\Theta_{-i}} \sup_{k_i \in K_i} v_i(k_i, \theta) \mu_{-i}(d\theta_{-i}|\theta_i), \\
\inf_{k_i \in K_i} \int_{\Theta_{-i}} v_i(k_i, \theta) \mu_{-i}(d\theta_{-i}|\theta_i) &= \int_{\Theta_{-i}} \inf_{k_i \in K_i} v_i(k_i, \theta) \mu_{-i}(d\theta_{-i}|\theta_i),
\end{align*}
\]

The converse is also true, i.e., for each bilateral mechanism \( \bar{k} \), there exists a collective mechanism \( k \) IPE to \( \bar{k} \).

**Remark 2** Since IPE can clearly preserve Bayesian incentive compatibility from collective mechanisms to bilateral mechanisms, this lemma also implies that all BIC bilateral mechanisms must be IPE to some BIC collective mechanisms under those hypotheses.

Lemma 1 relies on the boundary preserving property, which is satisfied in the quasi-separable environment, as an extension of the separable environment introduced in Chung and Ely (2006). It is applicable to a large class of economic scenarios and previous studies. In this quasi-separable environment, which is more straightforward to be checked, the boundary preserving property will be satisfied and IPE can be established. In the quasi-separable environment, interdependent valuations and correlated types are permitted. Each agent can separate his direct utility from his own contract and type and valuation adjustment from all agents’ types in a linear form of his payoff.

**Definition 6** A multi-agency contracting game is played in a **quasi-separable environment** if (1) \( K_i = \prod_{j=1}^{m_i} K_{ij} \) for each \( i \in \mathcal{N} \) and some \( m_i \in \mathbb{Z}^+ \). Its element is \( k_i = \prod_{j=1}^{m_i} k_{ij} \). Each \( K_{ij} \) is a connected, locally compact Polish space, and (2) for each \( i \in \mathcal{N}, j \in \{1, \cdots, m_i\} \), \( v_i(k, \theta) \equiv \sum_{j=1}^{m_i} h_{ij}(k_{ij}, \theta_i)w_{ij}(\theta) + q_i(\theta) \) for some continuous functions \( h_{ij} : K_{ij} \times \Theta_i \to \mathbb{R}, w_{ij} : \Theta \to \mathbb{R} \) satisfying either \( w_{ij}(\theta) \) is non-negative, or \( w_{ij}(\theta) \) is non-positive, and \( q_i : \Theta \to \mathbb{R} \). We call \( h_{ij}(k_{ij}, \theta_i) \) the **direct utility** from \( k_{ij} \) and \( \theta_i \), and call \( w_{ij}(\theta) \) the (interdependent) **valuation adjustment** of \( k_{ij} \).

\(^{21}\) Typical example of this space is any close or open subset of Euclidean space.
Then, in a quasi-separable environment we can identify a bilateral mechanism IPE to any collective mechanism.

**Proposition 3** In a quasi-separable environment, for each collective (respectively, collective BIC) mechanism $k$, there exists a bilateral (respectively, bilateral BIC) mechanism $\mathbf{k}$ IPE to $k$. The converse is also true, i.e., for each bilateral (respectively, bilateral BIC) mechanism $k$, there exists a collective (respectively, collective BIC) mechanism $\mathbf{k}$ IPE to $k$.

**Remark 3** If contract externalities are permitted, it is difficult to form a $k_j$ ($j \neq i$) coupled with $k_i$ such that $k$ IPE to $k$. The situation free of contract externalities raises the degree of freedom to find IPE bilateral mechanisms.

**Remark 4** Connectedness of the contract sets and quasi-separable form of the agents’ payoff functions are important. Consider a simple counterexample with finite case as follows. $N = \{1, 2\}$. $\mathcal{K}_1 = \{0, 1\}$. $\Theta_1$ is a singleton. $\Theta_2 = \{L, H\}$. $\theta_2$ are equally distributed. Let $v_1(0, L) = v_1(1, H) = 1$, and $v_1(1, L) = v_1(0, H) = 0$. Then consider $k_1$ such that $k_1(L) = 0$ and $k_1(H) = 1$. $\int_{\Theta_2} v_1(k_1(\theta), \theta) \mu_2(d\theta_2) = \frac{1}{2}(v_1(k_1(L), L) + v_1(k_1(H), H)) = 1$. But it is unlikely to find a (constant) bilateral $k_1(\theta_1) \in \mathcal{K}_1$ such that $\int_{\Theta_2} v_1(k_1(\theta_1), \theta) \mu_2(d\theta_2) = 1$.

Proposition 3 implies that in the quasi-separable environment the collection of bilateral mechanisms IPE to all collective BIC mechanisms is exactly the collection of all bilateral BIC mechanisms. In this respect, Proposition 3 completely characterizes collective BIC mechanisms via interim-payoff-equivalence with bilateral BIC mechanisms. Thus, the comparison of centralization and delegation boils down to the comparison between the highest payoff brought by collective BIC mechanisms and the highest payoff brought by the IPE bilateral mechanisms in the quasi-separable environment. This hints at a possibility of full equivalence from PL’s viewpoint.

### 6.2 An Illustrative Example for Interim-Payoff-Equivalent Mechanisms

Now consider a simple quasi-separable environment in the motivating vertical contracting example. The markets for different retailers to sell their goods are segmented. But interdependent valuations and correlated types are still permitted. Each retailer is a price taker on the downstream market. Thus, each retailer $i$ has a payoff function

$$v_i(x, t, \theta) = \pi_i(x_i, \theta) - t_i = P_i(\theta)x_i - t_i,$$

where $P_i$ is a positive continuous real-valued function of the market valuation signals $\theta$, which represents unit market price of $x_i$ on the downstream market for $A_i$. Then for any collective
BIC mechanism \((x, t)\), we can define each \(x_i : \Theta_i \to X_i\) by
\[
x_i(\theta_i) = \int_{\Theta_{-i}} \frac{P_i(\theta)}{P_i(\theta) \mu_{-i}(d\theta_{-i}|\theta_i)} x_i(\theta) \mu_{-i}(d\theta_{-i}|\theta_i),
\]
for each \(\theta_i\), and each \(\bar{t}_i : \Theta_i \to T_i\) by
\[
\bar{t}_i(\theta_i) = \int_{\Theta_{-i}} t_i(\theta) \mu_{-i}(d\theta_{-i}|\theta_i),
\]
for each \(\theta_i \in \Theta_i\). Easy to see \((\bar{x}, \bar{t})\) will be IPE to \((x, t)\) and become a bilateral BIC mechanism.

### 6.3 Full Equivalence Results

With additional assumptions on PL’s payoff function related to the agents’ payoff functions, Proposition 4 can usher in a few results on the full equivalence between collective and bilateral BIC mechanism designs (and therefore Bayesian menu design due to our delegation principle). The key for the full equivalence is to test whether the optimal BIC collective mechanism and at least one of bilateral BIC mechanisms IPE to collective BIC mechanisms can bring the same expected payoff to PL. In other words, we need to test whether
\[
\max_{k \text{ is BIC}} \int_{\Theta} u(k(\theta), \theta) \mu(d\theta) = \max_{k \text{ is IPE to } BIC} \int_{\Theta} u(k(\theta), \theta) \mu(d\theta).
\]
Nevertheless, our subsequent analysis will center on the conditions on the primitives for the full equivalence, and we need one more assumption below.

**Assumption 3** for each \(i \in \mathcal{N}\) the interim belief \(\mu_{-i}(\cdot|\cdot)\) is derived from the prior \(\mu\), that is, for any \(\mu\)-integrable functions \(\phi : \Theta \to \mathbb{R}\),
\[
\int_{\Theta} \phi(\theta) \mu(d\theta) = \int_{\Theta_i} \int_{\Theta_{-i}} \phi(\theta) \mu_{-i}(d\theta_{-i}|\theta_i) \mu_i(d\theta_i).
\]

Based on Proposition 3, if additionally PL’s payoff exhibits a certain relations with separate agents’ payoffs, the full equivalence can be ensured.

**Proposition 4** Under Assumption 3, in a quasi-separable environment, if for each \(i \in \mathcal{N}\), \(j \in \{1, \ldots, m_i\}\)

(i) \(u(k, \theta) \equiv \sum_{i=1}^{n} \sum_{j=1}^{m_i} G_{ij}(h_{ij}(k_{ij}, \theta_i)w_{ij}(\theta, \theta_i)) + L(\theta)\) for some continuous functions \(L : \Theta \to \mathbb{R}\) and \(G_{ij} : \mathbb{R} \times \Theta_i \to \mathbb{R}\) for each \(i, j\) satisfying \(G_{ij}(\cdot, \theta_i)\) is a concave transformation for each \(\theta_i\), and

(ii) \(\int_{\Theta_{-i}} G_{ij}(h_{ij}(k_{ij}, \theta_i)w_{ij}(\theta, \theta_i)) \mu_{-i}(d\theta_{-i}|\theta_i) = G_{ij}(\int_{\Theta_{-i}} h_{ij}(k_{ij}, \theta_i)w_{ij}(\theta) \mu_{-i}(d\theta_{-i}|\theta_i), \theta_i)\)

then for any optimal collective mechanism \(k^*\), there exists its IPE bilateral mechanism \(\bar{k}^*\) bringing to PL the same expected payoff. Thus, optimal bilateral BIC mechanism design is equivalent to optimal collective BIC mechanism design.

**Remark 5** It would be difficult to find conditions on the primitives for the exact equivalence between collective mechanisms and its IPE bilateral mechanisms if we allow non-separable re-
uation between the agents’ payoffs and the principal’s payoff. Because $k_i(\theta)$ and $k_{-i}(\theta)$ may simultaneously be integrated out with respect to $\theta_{-i}$ under $\mu_{-i}$.

Hypothesis (ii) in Proposition 4 can be further boiled down to two cases below.

**Proposition 5** In a quasi-separable environment, if for each $i \in \mathcal{N}$, $j \in \{1, \ldots, m_i\}$,

$$u(k, \theta) \equiv \sum_{i=1}^{n} \sum_{j=1}^{m_i} G_{ij}(h_{ij}(k_{ij}, \theta_i)w_{ij}(\theta), \theta_i) + L(\theta)$$

for some continuous functions $L : \Theta \to \mathbb{R}$ and $G_{ij} : \mathbb{R} \times \Theta_i \to \mathbb{R}$ for each $i$, $j$ satisfying $G_{ij}(., \theta_i)$ is a concave transformation for each $\theta_i$, then

$$\int_{\Theta_{-i}} G_{ij}(h_{ij}(k_{ij}, \theta_i)w_{ij}(\theta), \theta_i)\mu_{-i}(d\theta_{-i}|\theta_i) = G_{ij}(\int_{\Theta_{-i}} h_{ij}(k_{ij}, \theta_i)w_{ij}(\theta)\mu_{-i}(d\theta_{-i}|\theta_i), \theta_i)$$

if and only if for each $i \in \mathcal{N}$, $j \in \{1, \ldots, m_i\}$, and $\theta_i \in \Theta_i$, either

(i) $\int_{\Theta_{-i}} G_{ij}(h_{ij}(k_{ij}, \theta_i)w_{ij}(\theta), \theta_i)\mu_{-i}(d\theta_{-i}|\theta_i) = \int_{\Theta_{-i}} [a_{ij}(\theta_i)h_{ij}(k_{ij}, \theta_i)w_{ij}(\theta)]\mu_{-i}(d\theta_{-i}|\theta_i)$ for some continuous function $a_{ij} : \Theta_i \to \mathbb{R}$, or

(ii) $w_{ij}(\theta) \equiv w_{ij}(\theta_i)$.

Condition (i) in Proposition 5 implies PL’s payoff exhibits a certain linearly additive separability with the agents’ payoffs in the sense of interim expectations. It can clearly help preserve the interim-payoff equivalence and establish the full equivalence given interdependent valuations. Note that $a_{ij}$ can be either positive or negative. It is usually involved with partnership or social efficiency that $a'_{ij}$s take positive signs. In contrast, it reflects conflicts of interests between PL and the agents that $a'_{ij}$s take negative signs, especially in the traditional principal-agent relationship.

Condition (ii) in Proposition 5 permits non-linearly additive separability of PL’s payoff with the agents’ payoffs given private valuations. In some applications, PL’s payoff accordingly exhibits a certain non-linearly additive separability with each component as a concave transformation of each agent’s direct utility (given his own type), the full equivalence can be ensured.

Under either condition PL can predict a bilateral BIC mechanism interim-payoff-equivalent to the optimal collective BIC mechanism for her. Such bilateral mechanism will also be the optimal bilateral BIC mechanism. Then the full equivalence can be achieved. There are a few examples below in which Proposition 4 and condition (i) in Proposition 5 are applicable.

**Example 4** (Vertical Contracting 1) Recall the motivating (illustrative) example again. We additionally assume the manufacturer has the production cost as $b(\sum_{i=1}^{n} x_i)$ for some $b \in \mathbb{R}$. So her payoff is $u(t, x, \theta) \equiv \sum_{i=1}^{n} t_i - b(\sum_{i=1}^{n} x_i)$. Now we can set $a_{i1}(\theta_i) \equiv 1, a_{i2}(\theta_i) = \frac{b}{\int_{\Theta_{-i}} P_i(\theta)\mu_{-i}(d\theta_{-i}|\theta_i)}$. 
and \( L(\theta) \equiv 0 \), since

\[
\int_{\Theta_i} a_{ij}(\theta_i) P_i(\theta) x_i \mu_{-i}(d\theta_{-i} | \theta_i) = b x_i
\]

\[
= \int_{\Theta_i} b x_i \mu_{-i}(d\theta_{-i} | \theta_i),
\]

and \( a_{ii}(\theta_i) t_i = t_i \).

**Example 5** (Vertical Contracting 2) Now each retailer \( i \) is not a price taker on the downstream market. Simply assume he will face a downsloping inverse demand function \( P_i(x_i, \theta) = (\beta_i - \alpha_i x_i) w_i(\theta) \), where \( \beta_i > \alpha_i > 0 \), and \( w_i \) is nonnegative. Thus, each retailer \( i \) has a payoff function

\[
v_i(x_i, t_i, \theta) = \pi_i(x_i, \theta) - t_i
\]

\[
= x_i (\beta_i - \alpha_i x_i) w_i(\theta) - t_i.
\]

The manufacturer (PL) has a payoff \( u(t, x, \theta) \equiv \sum_{i=1}^{n} t_i - \sum_{i=1}^{n} c_i(x_i) \), where the cost function for each \( x_i \) is \( c_i(x_i) = \gamma_i(x_i - m_i)^2 \) for some \( \gamma_i, m_i > 0 \). If \( 2m_i \alpha_i = \gamma_i \beta_i \), we can set \( a_i(\theta_i) \equiv -\alpha_i \int_{\Theta_i} w_i(\theta) \mu_{-i}(d\theta_{-i} | \theta_i) \) and \( L(\theta) \equiv \gamma_i m_i^2 \).

**Example 6** (Ex Ante Efficient Allocation) Consider a classic resource allocation context. Contracts consist of the assignments of some divisible resources \( x = \prod_{i=1}^{n} x_i \in \mathbb{R}^n \) and the monetary transfers (from the agents to the social planner) \( t = \prod_{i=1}^{n} t_i \in \mathbb{R}^n \). Each agent \( i \) has a private evaluation \( \theta_i \) about the his assignment. His quasi-linear payoff function

\[
v_i(x_i, t, \theta) = w_i(\theta) h_i(x_i, \theta_i) - t_i.
\]

Then the planner’s ex post payoff function (social welfare) is \( \sum_{i=1}^{n} [w_i(\theta) h_i(x_i, \theta_i)] \), and she considers ex ante efficient allocation. Now we can set \( a_{ij}(\theta_i) \equiv 1 \) and \( L(\theta) \equiv 0 \).

**Example 7** (Teamwork) A headquarter assigns production tasks of a homogenous good to two downstream branches (indexed by \( i = 1, 2 \)). Each branch \( i \) has an efficiency parameter as its private type \( \theta_i \in \Theta_i \). The units of good the branch \( i \) produces is \( x_i \in [0, \infty) \). Contracts consist of the assignments of productions. Retailer \( i \) has a profit function \( w_i(\theta) h_i(x_i, \theta_i) \). The headquarter has a constant management cost \( c > 0 \), and then needs to maximize the full profit \( \sum_{i=1}^{n} w_i(\theta) h_i(x_i, \theta_i) - c \). Now we can set \( a_i(\theta_i) \equiv 1 \) and \( L(\theta) \equiv -c \).

**Example 8** (Outcome-contingent contracts) A principal assigns production tasks to two managers (indexed by \( i = 1, 2 \)). Each manager \( i \) has an efficiency parameter as its private type \( \theta_i \in [0, 1] \). Each manager \( i \) will yield an outcome \( \omega_i \geq 0 \) with a density function \( f_i(\omega_i, \theta) = \).
can draw from consumption of is the production cost of the quantity of the good purchased from (Procurement 1) A buyer (PL) needs procurement of two

Example 9 (Procurement 1) A buyer (PL) needs procurement of two imperfectly substitutive goods respectively from two producers indexed by $i \in \mathcal{N}$. $i$ receives a production cost signal $\theta_i \in [1,2]$. Contracts consist of the quantities of procurement $x = \prod_{i=1}^{n} x_i \in \prod_{i=1}^{n} [0, \infty)$, where $x_i$ is the quantity of the good purchased from $i$, and the monetary transfers $t = \sum_{i=1}^{n} t_i \in \prod_{i=1}^{n} [0, \infty)$, where $t_i$ is the monetary payment to $i$. Each producer $i$’s payoff is $t_i - c_i(x_i, \theta_i)$, where $c_i(x_i, \theta_i) = \theta_i x_i^2$ is the production cost of $x_i$. The buyer’s payoff is $\sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{n} t_i$. $\ln x_i$ is the payoff the buyer can draw from consumption of $x_i$. Now we can set $h_i(x_i, \theta_i) \equiv x_i^2$. Here define $G_i(-\theta_i x_i^2, \theta_i) = \frac{\ln(-\theta_i x_i^2)}{2} = \frac{\ln x_i^2}{2}$. $G_i(\cdot, \theta_i)$ is a concave transformation of $-\theta_i x_i^2$ given $\theta_i$.

Example 10 (Vertical contracting 3) Now assume each $x_i \in [1, \infty)$. Each retailer $i$ has the payoff $v_i(x, t, \theta) = w_i(\theta_i) P_i(x_i) x_i - t_i$, where $P_i$ and $w_i$ are positive continuous functions. $w_i(\theta_i) P_i(x_i)$ represents the inverse demand function parameterized by $\theta_i$ with respect to $x_i$. Let $P_i(x_i) = \frac{1}{x_i}$. The manufacturer’s payoff is $u(t, x, \theta) \equiv \sum_{i=1}^{n} t_i - (\sum_{i=1}^{n} b_i x_i^2)$, where $b_i x_i^2$ represents the manufacturing cost of producing $x_i$. Here define $G_i(w_i(\theta_i) x_i^{-2}, \theta_i) = -b_i \frac{w_i(\theta_i) x_i^{-2}}{w_i(\theta_i)} - 1$. Note that $-b_i x_i^2 = -b_i \left( \frac{w_i(\theta_i) x_i^{-2}}{w_i(\theta_i)} \right) - 1$. $G_i(\cdot, \theta_i)$ is a concave transformation of $w_i(\theta_i) x_i^{-2}$ given $\theta_i$.

7 Discussions

7.1 Individual Rationality Constraints

Our results can incorporate the participation constraints modeled as the individual rationality conditions. It can be formulated as follows. Suppose that $A_i$ has the commonly-observable reservation utility $r_i(\theta_i) \in \mathbb{R}$ based on his type $\theta_i$. A collective BIC mechanism $k$ is also (Bayesian) Individual Rational (IR) if for all $i \in \mathcal{N}$, $\theta \in \Theta$,

$$
\int_{\Theta \ni \theta} v_i(k(\theta), \theta) \mu_{-i}(d\theta_{-i} | \theta_i) \geq r_i(\theta_i). \quad (IR^c)
$$
A bilateral BIC mechanism $\tilde{k}$ is also (Bayesian) IR if for all $i \in \mathcal{N}$, $\theta \in \Theta$, 
\[
\int_{\Theta_{-i}} v_i(\tilde{k}(\theta), \theta) \mu_{-i}(d\theta_{-i}|\theta_i) \geq r_i(\theta_i). \quad (IR^b)
\]

A BNE contract selection profile $\tilde{k}$ is also (Bayesian) Individual Rational (IR) under a joint ex post menu $C$ if for each $i \in \mathcal{N}$, $\theta \in \Theta$, 
\[
\int_{\Theta_{-i}} v_i(\tilde{k}(\theta), \theta) \mu_{-i}(d\theta_{-i}|\theta_i) \geq r_i(\theta_i). \quad (IR^m)
\]

Such menu $C$ is hence said to be a (joint) Bayesian menu with IR constraints.

From the mathematical perspective, IR conditions serves similar to corresponding BIC conditions or BNE condition in the constraints of the multi-agency contracting problems. Apparently, given $\theta_i$, the right hand sides of the IR conditions (the interim payoffs under relevant mechanisms or contract selections) remain the same, and the left hand sides of the IR conditions ($r_i(\theta_i)$) are just some constants. Thus, it is not technically difficult to incorporate the individual rationality conditions in all the aforementioned results. In particular, interim payoff equivalence can still preserve Bayesian IR constraints.

### 7.2 Approximation of Full Equivalence

The value of our findings may also lie in approximation of full equivalence between bilateral and collective mechanism designs. It means that we can approximate optimal collective BIC mechanism design by optimal bilateral BIC mechanism design, even if the full equivalence does not exactly hold. For instance, although PL’s payoff function may have non-separable relation with respective agents’ payoff functions, linear approximation (first-order Taylor expansion) of PL’s payoff function in contracts can help to this end. Thus, our delegation principle and Proposition 3 may still be applicable to the linear approximation situation for the full equivalence in quasi-separable environments. Linear approximation can also measure the difference or approximation error between centralization and delegation with a bound. In fact, PL will still prefer delegation if such a bound is smaller than the practical cost of performing centralized contracting relative to delegated contracting.

Now consider a quasilinear quasi-separable environment. The agents’ types are as we assume in the general setting. The contract available to $A_i$ consists of assignment $x_i \in X_i$, where $X_i$ is a closed subset of $\mathbb{R}$, and transfer $t_i \in T_i$, where $T_i$ is a closed subset of $\mathbb{R}$. Write $x = (x_i)_{i \in \mathcal{N}}$, $t = (t_i)_{i \in \mathcal{N}}$, $X = \prod_{i=1}^n X_i$, and $T = \prod_{i=1}^n T_i$. $A_i$’s payoff is 
\[
v_i(x_i, t_i, \theta) = h_i(x_i, \theta)w_i(\theta) - t_i,
\]
where $h_i$ is a continuous real-valued function, and $w_i$ is a nonnegative continuous function. PL’s
payoff is

\[ u(x, t, \theta) = G(x, \theta) + \sum_{i=1}^{n} t_i, \]

where \( G \) is a continuous real-valued function. A collective mechanism is a list of Borel-measurable functions \((x, t) = ((x_i : \Theta \to X_i)_{i \in N}, t_i : \Theta \to T_i)_{i \in N}\). A bilateral mechanism is a list of Borel-measurable functions \((\mathbf{x}, \mathbf{t}) = ((\mathbf{x}_i : \Theta_i \to X_i)_{i \in N}, \mathbf{t}_i : \Theta_i \to T_i)_{i \in N}\).

Let \( U^* \) denote the optimal value of collective BIC mechanism design problem and \( U^{**} \) denote the optimal value of bilateral BIC mechanism design problem or Bayesian menu design problem.

**Definition 7** In the optimal mechanism, the optimal assignment rule \( \mathbf{x}^* \) (respectively, \( \mathbf{x}^* \)) is said to be regular if each \( \mathbf{x}_i^* \) (respectively, \( \mathbf{x}_i^* \)) is such that for each \( i \) and \( \theta, h_i(\mathbf{x}_i^* (\theta), \theta_i) \) (respectively, \( h_i(\mathbf{x}_i^* (\theta), \theta_i) \)) is integrable with respect to \( \mu \) over \( \theta \).

The regularity of optimal assignment rules can be achieved in many applications. Otherwise, \( G(\mathbf{x}^*(\cdot, \cdot)) \) or \( G(\mathbf{x}^*(\cdot, \cdot)) \) may not be integrable over \( \theta \) under \( \mu \). Especially, with the inclusion of individual rationality, such regularity may be more likely to hold. Under such regularities, we can obtain a result for approximation of full equivalence below.

**Proposition 6** Under Assumption 3, in the quasilinear quasi-separable environment, if

(i) \( G(x, \theta) = g((h_i(x_i, \theta_i))_{i \in N}, \theta) \), where \( g : \mathbb{R}^n \times \Theta \to \mathbb{R} \) is a function which is continuously second-order differentiable in each of first \( n \) arguments\(^{22}\) and continuous in \( \theta \). \( g_i \) denotes the partial derivative with respect to \( i \)-th argument, and \( g_{ij} \) denotes the \( ij \)-th second order derivative with respect to \( i \)-th and \( j \)-th arguments, for \( i, j \in N \),

(ii) for all \( i \) and \( \theta_i \), \( h_i(x_i^*, \theta_i) = 0 \) for some \( x_i^* \in X_i \),

(iii) for all \( i \) and \( \theta \), \( \int_{\Theta} g_i(0^{23}, \theta) \mu_{-i}(d\theta_{-i}|\theta_i) = \int_{\Theta} a_i(\theta_i)w_i(\theta)\mu_{-i}(d\theta_{-i}|\theta_i) \) for some continuous function \( a_i : \Theta_i \to \mathbb{R} \), and

(iv) for each \( i, j \), and \( \theta \), \( |g_{ij}((h_i(x_i, \theta_i))_{i \in N}, \theta)| \leq M \) for all \( x \),

then \( U^* - U^{**} \leq M \{ \sum_{i=1}^{n} \sum_{j=1}^{n} (\alpha_{ij} + \beta_{ij}) \} \) for some constants \( \alpha_{ij} \geq 0 \) and \( \beta_{ij} \geq 0 \) for all \( i \) and \( j \), when each \( x_i^* \) and \( \mathbf{x}_i^* \) are regular.

**Remark 6** \( g_{ij} \) denotes the degree of the strategic interdependence (either strategic complementarity or strategic substitutivity) between any two agents’ direct utilities from individual actions and types.\(^{24}\) The polar case is that \( g \) is linearly additive in all \( h_i \)’s. It actually represents strategic independence in PL’s payoff between any two individual agents’ actions, that is, each \( g_{ij} \) is equal to 0.

\(^{22}\)More rigorously, given \( \theta \), \( g \) is continuously second-order differentiable in the \( i \)-th argument over the interior of the range of \( h_i(\cdot, \theta_i) \), and left (respectively, right)-continuously-second-order-differentiable at the right (respectively, left) ending point of the range.

\(^{24}\)Generally speaking, this includes the "interdependence" between any individual agent’s direct utility and his direct utility themselves, which is reflected by second-order derivatives with respect to \( h_i \) itself.
Proposition 6 as an asymptotic result indicates that if \( G(x, \theta) \) can be expressed as a composite function of \( h_i(x_i, \theta_i)'s, \) and \( g_i(0, \theta) \) can be a linear transformation of \( \omega_i(\theta) \) by a multiplier \( a_i(\theta_i) \) in the sense of interim expectations, optimal bilateral mechanism design is approaching optimal collective mechanism design in PL’s viewpoint through IPE, as the degrees of the strategic interdependence approach zero. Moreover, the loss of bilateral mechanism design may increase as the number of agents increases, since each \( \alpha_{ij} \) and \( \beta_{ij} \) must be nonnegative. Here are several examples to which Proposition 6 can apply.

Example 11 (Vertical Contracting 4) Recall the illustrative example in terms of Example 4, which fits the assumption of quasilinear quasi-separable environment. But now the supplier’s payoff takes a non-separable form in \( x, \) \( \sum_{i=1}^{n} t_i - c(x) \), where \( c(x) = \frac{1}{2}(\sum_{i=1}^{n} x_i)^2 \) is the cost function of the supplier. For each \( i, \) \( c_i(x) = \sum_{i=1}^{n} x_i. \) Compared to Proposition 6, \( h_i(x_i, \theta_i) \equiv x_i \) here. \( c_i(0) = 0. \) So we can choose \( a_i(\theta_i) \equiv 0. \) For each \( i,j, c_{ij}(x_1, x_2) = 1. \) Then Proposition 6 implies \( U^* - U^{**} \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} + \beta_{ij}. \)

Example 12 (Procurement 2) A buyer (PL) needs procurement of two imperfectly substitutive goods respectively from \( n \) producers indexed by \( i = 1, 2. \) \( i \) receives a production cost signal \( \theta_i > 0. \) Contracts are the same as in example 8 of Procurement 1. Each producer \( i \)'s payoff is \( t_i - c_i(x_i, \theta_i), \) where \( c_i(x_i, \theta_i) = \theta_i x_i^2 \) is the production cost of \( x_i. \) The buyer’s payoff is \( B(x) - \sum_{i=1}^{n} t_i, \) where \( B(x) = -e^{-(x_1^2+x_2^2)} \) is the benefit the buyer can draw from consumption of \( x. \) Let \( B(x) = G(x_1^2, x_2^2), \) \( G_i(x_1^2, x_2^2) = e^{-(x_1^2+x_2^2)}, \) and \( G_i(0) = 1. \) We can choose \( a_i(\theta_i) \equiv \frac{1}{\theta_i}. \) Moreover, \( G_{ij}(x_1^2, x_2^2) = -e^{-(x_1^2+x_2^2)}. \) Clearly, \( |G_{ij}(x_1^2, x_2^2)| \leq 1. \) Then Proposition 6 implies \( U^* - U^{**} \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij}, \) where \( \alpha_{ij} = \max_{x_i \in X_i, x_j \in X_j} x_1^2 x_2^2. \)

Example 13 (Resources Allocation) A social planner allocates a homogenous good to two agents (indexed by \( i = 1, 2. \)) The units of good agent \( i \) receives is \( x_i \in [0, \overline{x}_i]. \) The monetary transfer from \( i \) to the planner is \( t_i \in [0, t_i]. \) Each agent \( i \) has a private type \( \theta_i \in [0, \Theta_i) \) and then a payoff function \( w_i(\theta)x_i - t_i. \) But such good has some negative externality effect \( C(x) = (\sum_{i=1}^{n} x_i)^2. \) The planner’s payoff is the social surplus \( G(x, \theta) = \sum_{i=1}^{n} w_i(\theta_i)x_i - (\sum_{i=1}^{n} x_i)^2. \) Given \( \theta, \) \( G_i(x, \theta) = w_i(\theta) - 2(x_1 + x_2) \) and \( G_i(0, \theta) = w_i(\theta). \) For each \( i,j, \) \( G_{ij}(x_1, x_2) = -2. \) Then Proposition 6 implies \( U^* - U^{**} \leq 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij}, \) where \( \alpha_{ij} = \max_{x_i \in X_i, x_j \in X_j} x_1 x_2. \)

7.3 A Characterization Example for Full Equivalence

In the quasilinear quasi-separable environment, canonical results show that the optimal collective BIC mechanisms can be characterized. We now further discuss the motivating example to show how its IPE mechanism will lead to full equivalence based on characterization.

Note that the motivating example is actually provided in a quasilinear quasi-separable environment. We additionally assume the types of different agents are independently distributed.
Each $\theta_i \in \Theta_i$ is drawn from a cumulative probability distribution $F_i(\theta_i)$ with $\Theta_i = [\underline{\theta}_i, \overline{\theta}_i]$. The joint cumulative probability distribution for $\theta$ is $F(\theta)$. Bayesian updated distribution becomes $F_{-i}(\theta_{-i})$. $P_i(\theta)$ is differentiable with respect to each $\theta_i$. Let $\Lambda_i$’s reservation utility $r_i(\theta_i) \equiv 0$. The rest setting is same as in Example 4.

In the similar argument of Myerson (1981), we can characterize the optimal collective BIC and IR mechanism $(\mathbf{x}^*, t^*)$ to be such that

$$
\mathbf{x}^* \in \arg\max_{\mathbf{x}} \int_{\Theta} \left\{ -b \left( \sum_{i=1}^{n} x_i(\theta) \right) + \sum_{i=1}^{n} \left( P_i(\theta) x_i(\theta) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial P_i(\theta)}{\partial \theta_i} x_i(\theta) \right) \right\} dF(\theta)
$$

s.t. $x_i(\theta)$ is increasing in $\theta_i$, for each $i$,

and each $t_i^*(\theta) = \int_{\overline{\theta}_i}^{\theta_i} \frac{\partial P_i(\theta)}{\partial \theta_i} x_i^*(s_{-i}) ds_{-i} - P_i(\theta) x_i^*(\theta)$. Moreover, the optimal objective value of PL is equal to

$$
\int_{\Theta} \left\{ -b \left( \sum_{i=1}^{n} x_i^*(\theta) \right) + \sum_{i=1}^{n} \left( P_i(\theta) x_i^*(\theta) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial P_i(\theta)}{\partial \theta_i} x_i^*(\theta) \right) \right\} dF(\theta)
$$

$$
= \int_{\Theta} \left\{ \sum_{i=1}^{n} \left[ (P_i(\theta) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial P_i(\theta)}{\partial \theta_i} - b)x_i^*(\theta) \right] \right\} dF(\theta).
$$

Recall the bilateral mechanism $(\mathbf{x}^*, \mathbf{t}^*)$ IPE to $(\mathbf{x}^*, t^*)$ is such that

$$
\mathbf{x}^*_i(\theta_i) = \int_{\Theta_{-i}} \int_{\Theta_{-i}} \frac{P_i(\theta)}{P_i(\theta) \mu_{-i}(d\theta_{-i}|\theta_i)} x_i^*(\theta) dF_{-i}(\theta_{-i}),
$$

(1)

for each $\theta_i$, and each $\mathbf{t}_i : \Theta_i \rightarrow T_i$ by

$$
\mathbf{t}^*_i(\theta_i) = \int_{\Theta_{-i}} t_i^*(\theta) \mu_{-i}(d\theta_{-i}|\theta_i),
$$

for each $\theta_i \in \Theta_i$. Clearly, $\mathbf{x}^*_i(\theta_i)$ is also increasing in $\theta_i$. Moreover, by plugging (4), easy to check

$$
\int_{\Theta} \left\{ \sum_{i=1}^{n} \left[ (P_i(\theta) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial P_i(\theta)}{\partial \theta_i} - b)x_i^*(\theta) \right] \right\} f(\theta) d\theta
$$

$$
= \int_{\Theta} \left\{ \sum_{i=1}^{n} \left[ (P_i(\theta) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial P_i(\theta)}{\partial \theta_i} - b)x_i^*(\theta) \right] \right\} f(\theta) d\theta.
$$

Thus, the full equivalence does hold in this context.

---

25 So $(\mathbf{x}^*, \mathbf{t}^*)$ must be BIC and IR as well.
7.4 Primitive Constraints across the Contracts for Different Agents

The aforementioned results do not address the explicit primitive constraints across the contracts for different agents under which $K$ is not directly equal to the product of the agents’ contract sets, that is, $K \subseteq \prod_{i=1}^{n} K_i$, but $K \neq \prod_{i=1}^{n} K_i$. For instance, in auction mechanism design, the sum of the probability assignments for different agents must not be greater than 1.

It is not technically difficult to find our delegation principle also holds in this generalized situation with $C^* = \prod_{i=1}^{n} \text{cl}\{((K_i^* (\theta_i) : \theta_i \in \Theta_i) \cap K$ instead in part (i) of Proposition 2. Meanwhile, if the IPE bilateral BIC mechanism $k^*$ still satisfies the primitive constraint, i.e., $k^*(\theta) \in K$ for each $\theta$, Proposition 3 will still hold. Even as long as the BIC bilateral mechanism IPE to optimal collective mechanism $k^*$ still satisfies the primitive constraint, the full equivalence results still hold. Here is a simple example for that.

**Example 14** (Procurement 3) One producer procures two input goods separately from two input suppliers denoted by $i = 1, 2$. Contracts are the same as in example 9 of Procurement 1. $\theta_i$’s are independently distributed. $i$’s payoff function $v_i(t_i, x_i, \theta_i) = t_i - \theta_i x_i$. The producer’s payoff $u(x, t, \theta) = x_1^{\alpha} x_2^{1-\alpha} - t_1 - t_2$. $x_1^{\alpha} x_2^{1-\alpha}$ denotes the Cobb-Douglas (monetary) production function, where $\alpha \in (0, 1)$. There is a constraint over $x_i$’s: $x_1^{\alpha} x_2^{1-\alpha} \leq \bar{q}$, where $\bar{q}$ denotes the capacity limit. The producer should not purchase the bundle of $(x_1, x_2)$ beyond the production capacity constraint. For each $i$, collective (respectively bilateral) BIC assignment rule for $i$ is $x_i: \Theta \rightarrow \mathbb{R}$ (respectively, $\mathfrak{x}_i: \Theta_i \rightarrow \mathbb{R}$). Suppose optimal collective BIC assignment rule is $x^*$. Then its IPE bilateral assignment rule $\mathfrak{x}^*$ can be defined by

$$\mathfrak{x}_i^*(\theta_i) = \int_{\Theta_{-i}} x_i^*(\theta) \mu_{-i}(d\theta_{-i}), i = 1, 2.$$  

Clearly, if for each $\theta$, $x_1^*(\theta)x_2^{1-\alpha}(\theta) \leq \bar{q}$, then $\mathfrak{x}_1^*(\theta_1)\mathfrak{x}_2^{1-\alpha}(\theta_2) \leq \bar{q}$.

However, in some cases IPE bilateral mechanisms may not necessarily preserve some primitive constraints across the contracts for different agents, especially for those linear combination inequality constraints, such as the natural requirement on probabilistic assignments in auction design. It is generally difficult to provide conditions on the primitives for such preservation. Such preservation may need some requirement on the properties of the optimal collective mechanism per se. For instance, the symmetric mechanism design with ex ante identical agents may help to this end, especially in auction contexts.

---

26 Thus, the subgames induced by preoffered menus are related to the "generalized games" or "constrained games" introduced by Arrow and Debreu (1954), and Rosen (1965).
8 Conclusion

We modelize and analyze the substantial analytical rationale of bilateral Bayesian mechanism design in multi-agency contracting to show how useful bilateral mechanisms turn out to be as the practical contracting procedure. Bilateral Bayesian mechanism can be equivalently translated to Bayesian menu design, which is even simpler to perform in formality of contracting execution. Even if available mechanisms can be collective, Bayesian updated beliefs as common knowledge and interim-payoff-equivalence can help provide the possibility to establish the full equivalence between bilateral and collective mechanism designs. One can benefit from the finer information structure in Bayesian implementation relative to ex post (or dominant-strategy) implementation to deal with information asymmetry. Even if the full equivalence does not exactly hold, its approximation may still be available for further scrutiny. These facts make bilateral mechanism design theoretically reasonable for practical decision makers in real life. Our analysis may have broad economic applications, including nonlinear pricing, resource allocation, regulation, insurance, public choice, etc. It could also be more tractable to establish the full equivalence under more concrete application environments. In large contracting games, since every agent becomes negligible somehow, the full equivalence is worth further discussion as well. Our approximation results may also be useful in the empirical study of mechanism design.

Appendix

Lemma 2 For any \( C \in P_f(K) \) satisfying \( \Psi(\cdot, C) \) is well-defined, \( \Psi(\cdot, C) \) is a closed-valued Borel-measurable set valued function from \( \Theta \) to \( C \).

Proof. First claim that \( \Psi(\cdot, C) \) has a closed graph for any \( C \in P_f(K) \) satisfying \( \Psi(\cdot, C) \) is well-defined. For simplicity, let \( F_C(\cdot) = \Psi(\cdot, C) \). We need to show that \( GrF_C = \{ (\theta, \tilde{k}(\theta)) \in \Theta \times C | \tilde{k} \) is the BNE under \( C \} \) is closed.

First fix \( \theta \in \Theta \). Pick any arbitrary sequence \( \{ (\theta^l, \tilde{k}(\theta^l)) \} \) in \( GrF_C \) satisfying

\[
\tilde{k}(\theta^l) \in F_C(\theta^l), \text{ and } (\theta^l, \tilde{k}(\theta^l)) \to (\theta, \tilde{k}(\theta)), \text{ as } l \to \infty.
\]

Thus it suffices to show that \( \tilde{k}(\theta) \in F_C(\theta) \), that is, for each \( i \in N \),

\[
\int_{\Theta_{-i}} v_i(\tilde{k}(\theta), \theta)\mu_{-i}(d\theta_{-i}|\theta_i) \geq \int_{\Theta_{-i}} v_i(\tilde{k}^l_i(\theta_i), \tilde{k}_{-i}(\theta_{-i}), \theta)\mu_{-i}(d\theta_{-i}|\theta_i),
\]

for all \( \tilde{k}^l_i \in F_i \) satisfying \( \tilde{k}^l_i(\theta_i) \in C_i \).

For each \( i \in N \),

\[
\int_{\Theta_{-i}} v_i(\tilde{k}(\theta^l), \theta^l)\mu_{-i}(d\theta_{-i}|\theta_i^l) \geq \int_{\Theta_{-i}} v_i(k_i^l(\theta_i^l), k_{-i}^l(\theta_{-i}^l), \theta^l)\mu_{-i}(d\theta_{-i}|\theta_i^l),
\]
for all \( k_i' \in F_i \) satisfying \( \tilde{k}_i'(\theta_i) \in C_i \). Since \( v_i \) is continuous and is bounded on \( \Theta \) given \( k \) by the primitives and Assumption 1, Delbaen's Lemma (1974)\(^{27}\) implies that
\[
\int_{\Theta^{-i}} v_i(\tilde{k}(\theta), \theta) \mu^{-i}(d\theta^{-i}|\theta_i) \geq \int_{\Theta^{-i}} v_i(\tilde{k}_i'(\theta_i), \tilde{k}_{-i}(\theta_{-i}), \theta) \mu^{-i}(d\theta^{-i}|\theta_i),
\]
for all \( k_i' \in F_i \) satisfying \( \tilde{k}_i'(\theta_i) \in C_i \). Therefore, the graph of \( \Psi(\cdot, C) \) is closed in \( \Theta \times C \), i.e., \( \Psi(\cdot, C) \) is closed-valued.

Moreover, \( \Theta, K \), and \( C \) are all Polish. Thus, by Theorem 3.5 in Himmelberg (1975), \( \Psi(\cdot, C) \) is Borel-measurable. \( \square \)

**Proof of Proposition 1.**

(i). Assume that \( \tilde{k} \in \mathcal{F}(\Theta, K) \) is BIC. Define
\[
C = \prod_{i=1}^n \text{cl}\{ (\tilde{k}_i(\theta_i) : \theta_i \in \Theta_i ) \}.
\]

First claim that \( \tilde{k}(\theta) \in \Psi(\theta, C) \) for all \( \theta \in \Theta \), that is, for each \( i \in N \), and each \( \theta_i \in \Theta_i \),
\[
\int_{\Theta^{-i}} v_i(\tilde{k}(\theta), \theta) \mu^{-i}(d\theta^{-i}|\theta_i) \geq \int_{\Theta^{-i}} v_i(\tilde{k}_i'(\theta_i), \tilde{k}_{-i}(\theta_{-i}), \theta) \mu^{-i}(d\theta^{-i}|\theta_i),
\]
for all \( k_i' \in F_i \) satisfying \( (\tilde{k}_i'(\theta_i), \tilde{k}_{-i}(\theta_{-i})) \in C \). Suppose not. Then for some agent \( j \), some \( \theta_j' \in \Theta_j \), and some \( k_j' \in F_j \) satisfying \( (k_j'(\theta_j'), \tilde{k}_{-j}(\theta_{-j})) \in C \),
\[
\int_{\Theta^{-j}} v_j(\tilde{k}_j(\theta'), \tilde{k}_{-j}(\theta_{-j}), \theta_j', \theta_{-j}) \mu_{-j}(d\theta_{-j}|\theta_j') < \int_{\Theta^{-j}} v_j(k_j'(\theta_j'), \tilde{k}_{-j}(\theta_{-j}), \theta_j', \theta_{-j}) \mu_{-j}(d\theta_{-j}|\theta_j').
\]
Because of the definition of \( C \), any section of \( C \) is still closed. Thus, for any \( \theta_{-j} \), there exists a sequence of type \( \{ \theta_{j,l} \}_l \) in \( \Theta_j \) such that \( (\tilde{k}_j(\theta_{j,l}), \tilde{k}_{-j}(\theta_{-j})) \to (k_j'(\theta_j'), \tilde{k}_{-j}(\theta_{-j})) \) in \( C \), as \( l \to \infty \). Hence, by the continuity of \( v_j \) and Delbaen’s Lemma (1974), for \( l \) large enough, (2) implies
\[
\int_{\Theta^{-j}} v_j(\tilde{k}_j(\theta'), \tilde{k}_{-j}(\theta_{-j}), \theta_j', \theta_{-j}) \mu_{-j}(d\theta_{-j}|\theta_j') < \int_{\Theta^{-j}} v_j(k_j'(\theta_j'), \tilde{k}_{-j}(\theta_{-j}), \theta_j', \theta_{-j}) \mu_{-j}(d\theta_{-j}|\theta_j').
\]
This contradicts the fact that \( \tilde{k} \) is BIC and proves the claim.

Thus, \( \Psi(\cdot, C) \) is clearly well-defined. By Lemma 1, \( \Psi(\cdot, C) \) is Borel-measurable. Therefore, \( \tilde{k} \) is actually a Borel-measurable selection from \( \Psi(\cdot, C) \).

(ii). Assume that \( \tilde{k}(\theta) \in \Psi(\theta, C) \subseteq C \) for all \( \theta \in \Theta \). For all \( i \in N \), all \( \theta_i \in \Theta_i \), and all
\(^{27}\) Another description about this lemma can be found in Page (1987)
$k'_i \in \mathcal{F}_i$ satisfying $(k'_i(\theta_i), k_{-i}(\theta_{-i})) \in C$, 

$$\int_{\Theta_{-i}} v_i(\bar{k}(\theta), \theta) \mu_{-i}(d\theta_{-i}|\theta_i) \geq \int_{\Theta_{-i}} v_i(k'_i(\theta_i), \bar{k}_{-i}(\theta_{-i}), \theta) \mu_{-i}(d\theta_{-i}|\theta_i).$$

Since there is always some $k'_i$ satisfying $k'_i(\theta_i) = \bar{k}(\theta_i)$ for any $\theta'_i \in \Theta_i$, we have 

$$\int_{\Theta_{-i}} v_i(\bar{k}(\theta), \theta) \mu_{-i}(d\theta_{-i}|\theta_i) \geq \int_{\Theta_{-i}} v_i(\bar{k}(\theta'_i), \bar{k}_{-i}(\theta_{-i}), \theta) \mu_{-i}(d\theta_{-i}|\theta_i),$$

for all $\theta'_i \in \Theta_i$. Thus, $\bar{k}$ is BIC. □

**Remark on Proposition 1.**

Furthermore, when $\Psi(\cdot, C)$ is well-defined for some $C \in P_f(K)$, the Bayesian menu design problem (P2) can be rewritten in a compact way:

$$\max_{C \in P_f(K)} \int_{\Theta} \max_{\bar{k}(\theta) \in \Psi(\theta, C)} u(\bar{k}(\theta), \theta) \mu(d\theta).$$

The feasible bilateral BIC mechanism set is defined as

$$\mathcal{IC}^b = \{ \bar{k} \in \mathcal{F}(\Theta, K) : \bar{k} \text{ is BIC} \}.$$  

The bilateral BIC mechanism design problem (P1') can also be stated compactly as

$$\max_{\bar{k} \in \mathcal{IC}^b} \int_{\Theta} u(\bar{k}(\theta), \theta) \mu(d\theta).$$

Moreover, the equivalent mechanism set induced by a joint menu $C \in P_f(K)$ is defined by

$$\Sigma_\Psi(C) = \{ \bar{k} \in \mathcal{F}(\Theta, K) : \bar{k}(\theta) \in \Psi(\theta, C) \text{ for all } \theta \in \Theta \}. \quad (3)$$

It denotes the set of all measurable selections from $\Psi(\cdot, C)$ in $\mathcal{F}(\Theta, K)$ for a given menu $C \in P_f(K)$. Next, the full equivalent mechanism set induced by all joint menus is defined by

$$\Sigma_\Psi = \bigcup_{C \in P_f(K)} \Sigma_\Psi(C).$$

Indeed, Proposition 1 is equivalent to say $\mathcal{IC}^b = \Sigma_\Psi$. □

**Lemma 3** For each $C \in P_f(K)$ satisfying $\Psi(\cdot, C)$ is well-defined, there exists some $\bar{k} \in \Sigma_\Psi(C)$ such that

$$u(\bar{k}(\theta), \theta) = \max_{\bar{k}(\theta) \in \Psi(\theta, C)} u(\bar{k}(\theta), \theta),$$
for all \( \theta \in \Theta \). Moreover, the function \( \theta \mapsto \max_{k \in \Psi(\theta, C)} u(k, \theta) \) is Borel measurable.

**Proof.** Note that \( \Theta \) and \( \mathcal{K} \) are Borel space. By Lemma 2, for each \( C \in P_f(\mathcal{K}) \) satisfying \( \Psi(\cdot, C) \) is well-defined, \( \Psi(\cdot, C) \) is Borel-measurable and compact-valued. We know \( u \) is Borel-measurable and \( u(\cdot, \theta) \) is continuous. Then by Theorem 2 in Himmelberg, Parthasarathy and Van Vleck (1976), there exists some Borel measurable selector \( \tilde{k} \) in \( \Sigma_\Psi(C) \) for the set-valued function \( \Psi(\theta, C) \) such that \( u(\tilde{k}(\theta), \theta) = \max_{\tilde{k}(\theta) \in \Psi(\theta, C)} u(k(\theta), \theta) \) for all \( \theta \in \Theta \). Moreover, the function \( \theta \mapsto \max_{\tilde{k}(\theta) \in \Psi(\theta, C)} u(k(\theta), \theta) \) is also Borel measurable. □

**Remark.** (Theorem 2 in Himmelberg, Parthasarathy and Van Vleck (1976))

Let \( S \) and \( A \) be Borel spaces, let \( F \) be a Borel measurable compact valued multifunction from \( S \) to \( A \), and let \( u : GrF \to \mathbb{R} \) be a Borel measurable function such that \( u(s, \cdot) \) is an upper-semicontinuous function on \( F(s) \) for each \( s \in S \). Then there exists a Borel measurable selector \( f : S \to A \) for \( F \) such that

\[
u(s, f(s)) = \max_{a \in F(s)} u(s, a)
\]

for all \( s \in S \). Moreover, the function \( v \) defined by \( v(s) = \max_{a \in F(s)} u(s, a) \) is Borel measurable.

**Proof of Proposition 2.**

(i). By the proof of Proposition 1, \( k^*(\theta) \in \Psi(\theta, C^*) \), we have

\[
\int_{\Theta} \max_{\tilde{k}(\theta) \in \Psi(\theta, C^*)} u(\tilde{k}(\theta), \theta) \mu(d\theta) \geq \int_{\Theta} u(k^*(\theta), \theta) \mu(d\theta).
\]

Thus, for all \( k \in \mathcal{IC}^b \),

\[
\int_{\Theta} u(k^*(\theta), \theta) \mu(d\theta) \geq \int_{\Theta} u(k(\theta), \theta) \mu(d\theta).
\]

Then, by Proposition 1, \( \mathcal{IC}^b = \bigcup_{C \in P_f(\mathcal{K})} \Sigma_\Psi(C) \). Hence, for all \( k \in \bigcup_{C \in P_f(\mathcal{K})} \Sigma_\Psi(C) \), we have

\[
\int_{\Theta} u(k^*(\theta), \theta) \mu(d\theta) \geq \int_{\Theta} u(k(\theta), \theta) \mu(d\theta).
\]

(4)

Moreover, by Lemma 3, for each \( C \in P_f(\mathcal{K}) \), there exists some \( k' \in \Sigma_\Psi(C) \) such that \( u(k'(\theta), \theta) = \max_{\tilde{k}(\theta) \in \Psi(\theta, C)} u(\tilde{k}(\theta), \theta) \) for all \( \theta \in \Theta \). Thus, by (3), for each \( C \in P_f(\mathcal{K}) \), we have

\[
\int_{\Theta} u(k'(\theta), \theta) \mu(d\theta) \geq \int_{\Theta} \max_{\tilde{k}(\theta) \in \Psi(\theta, C)} u(\tilde{k}(\theta), \theta) \mu(d\theta).
\]

Therefore,

\[
\int_{\Theta} \max_{\tilde{k}(\theta) \in \Psi(\theta, C^*)} u(\tilde{k}(\theta), \theta) \mu(d\theta) \geq \int_{\Theta} \max_{\tilde{k}(\theta) \in \Psi(\theta, C)} u(\tilde{k}(\theta), \theta) \mu(d\theta)
\]
for all \( C \in P_f(K) \). Hence, \( C^* \) solves (P3).

Clearly,
\[
\max_{C \in P_f(K)} \int \max_{\tilde{k}(\theta) \in \Psi(\theta, C)} u(\tilde{k}(\theta), \theta) \mu(d\theta) = \int \max_{\tilde{k}(\theta) \in \Psi(\theta, C^*)} u(\tilde{k}(\theta), \theta) \mu(d\theta)
\]

(ii). By Lemma 3, for each \( C \in P_f(K) \), there always exists some \( \tilde{k} \) such that
\[
uu(\tilde{k}(\theta), \theta) = \max_{\tilde{k}(\theta) \in \Psi(\theta, C^*)} u(\tilde{k}(\theta), \theta)
\]
for any \( \theta \in \Theta \). In particular, now consider \( \tilde{k} \in \Sigma_\Psi(C^*) \) satisfying \( \tilde{k}(\theta) \in \arg\max_{\tilde{k}(\theta) \in \Psi(\theta, C^*)} u(\tilde{k}(\theta), \theta) \).

for all \( \theta \in \Theta \). For each \( C \in P_f(K) \), by hypotheses,
\[
\int \max_{\tilde{k}(\theta) \in \Psi(\theta, C)} u(\tilde{k}(\theta), \theta) \mu(d\theta) = \int \max_{\tilde{k}(\theta) \in \Psi(\theta, C^*)} u(\tilde{k}(\theta), \theta) \mu(d\theta)
\]

for all \( \tilde{k} \in \Sigma_\Psi(C) \) satisfying \( u(\tilde{k}(\theta), \theta) = \max_{\tilde{k}(\theta) \in \Psi(\theta, C^*)} u(\tilde{k}(\theta), \theta) \) for any \( \theta \in \Theta \). It implies that
\[
\int \max_{\tilde{k}(\theta) \in \Psi(\theta, C)} u(\tilde{k}(\theta), \theta) \mu(d\theta) = \max_{\tilde{k}(\theta) \in \Psi(\theta, C^*)} \int \max_{\tilde{k}(\theta) \in \Psi(\theta, C)} u(\tilde{k}(\theta), \theta) \mu(d\theta).
\]

Also, by Proposition 1,
\[
\Sigma_\Psi = \bigcup_{C \in P_f(K)} \Sigma_\Psi(C).
\]

Hence, by (4) and (5), we have
\[
\max_{C \in P_f(K)} \int \max_{\tilde{k}(\theta) \in \Psi(\theta, C)} u(\tilde{k}(\theta), \theta) \mu(d\theta) = \max_{\tilde{k}(\theta) \in \Psi(\theta, C^*)} \int u(\tilde{k}(\theta), \theta) \mu(d\theta)
\]

Therefore, \( \tilde{k}^* \) solves (P2). \( \square \)

Proof of Lemma 1.

"if". For each \( i \), given \( k_i, \theta_i \), define \( \Phi_i : K_i \rightarrow \mathbb{R} \) by \( \Phi_i(k_i) = \int_{\Theta_{-i}} v_i(k_i, \theta) \mu_{-i}(d\theta_{-i}|\theta_i) \). Since \( v_i \) is continuous, \( \Phi_i \) is also continuous. Thus, connectedness of \( K_i \) implies that the range of \( \Phi_i \) is also connected in \( \mathbb{R} \) and therefore should be a interval \( I_{i, \theta_i} \) taking the form of \([a_i(\theta_i), b_i(\theta_i)]\),
[a_i(\theta_i), b_i(\theta_i)), (a_i(\theta_i), b_i(\theta_i))], and (a_i(\theta_i), b_i(\theta_i))$. Moreover, $\Phi_i$ is clearly onto from $K_i$ to $I_{i,\theta_i}$.

Then we define the inverse set-valued function of $\Phi_i$ as $\rho_i^{-1}_{\theta_i}: I_{i,\theta_i} \rightarrow K_i$ by

$$\rho_i^{-1}_{\theta_i}(x) = \{k_i \in K_i : \Phi_i(k_i) = x\}$$

By the a variant of closed map lemma, local compactness of $K$ implies that $\Phi_i$ is a continuous closed map. Thus, $\rho_i^{-1}_{\theta_i}$ must be nonempty closed-valued and is a measurable set-valued function. Kuratowski-Ryll-Nardzewski Selection Theorem implies that $\rho_i^{-1}_{\theta_i}$ must admit a Borel-measurable selector, say $\varphi_i, \theta_i: I_{i,\theta_i} \rightarrow K_i$.

Now define $\phi_{i,k_i}: \Theta_i \rightarrow \mathbb{R}$ by

$$\phi_{i,k_i}(\theta_i) = \int_{\Theta_{-i}} v_i(k_i(\theta), \theta) \mu_{-i}(d\theta_{-i}|\theta_i).$$

Obviously, $\phi_{i,k_i}$ is a Borel-measurable function of $\theta_i$ due to the assumption on $\mu_{-i}$ in the model primitives.

Moreover, for all $\theta$, $v_i(k_i(\theta), \theta)$ will be contained in an interval taking the same form as $I_{i,\theta_i}$ but with two boundary points as $\inf_{k_i \in K_i} v_i(k_i, \theta)$ and $\sup_{k_i \in K_i} v_i(k_i, \theta)$. Thus, $\phi_{i,k_i}(\theta_i)$ will be contained in another interval taking the same form as $I_{i,\theta_i}$ but with two boundary points as $\inf_{k_i \in K_i} v_i(k_i, \theta) \mu_{-i}(d\theta_{-i}|\theta_i)$ and $\sup_{k_i \in K_i} v_i(k_i, \theta) \mu_{-i}(d\theta_{-i}|\theta_i)$.

Hypothesis (2) implies that

$$\int_{\Theta_{-i}} \sup_{k_i \in K_i} v_i(k_i, \theta) \mu_{-i}(d\theta_{-i}|\theta_i) = \sup_{k_i \in K_i} \int_{\Theta_{-i}} v_i(k_i, \theta) \mu_{-i}(d\theta_{-i}|\theta_i) = b_i(\theta_i),$$

and

$$\int_{\Theta_{-i}} \inf_{k_i \in K_i} v_i(k_i, \theta) \mu_{-i}(d\theta_{-i}|\theta_i) = \inf_{k_i \in K_i} \int_{\Theta_{-i}} v_i(k_i, \theta) \mu_{-i}(d\theta_{-i}|\theta_i) = a_i(\theta_i).$$

Note that $\phi_{i,k_i}(\theta_i) \in I_{i,\theta_i}$ for all $\theta_i$ if and only if (1) or (2) holds.

Therefore, we can define a function $\bar{k}_i: \Theta_i \rightarrow K_i$ by

$$\bar{k}_i(\theta_i) = \varphi_{i,\theta_i}(\phi_{i,k_i}(\theta_i)), \text{ for each } \theta_i \in \Theta_i.$$ 

Hence $\bar{k}_i$ is clearly a Borel-measurable function, and then $\bar{k}$ is a well-defined bilateral mechanism.

Next by the definitions above, for each $\theta_i \in \Theta_i$, easy to see

$$\int_{\Theta_{-i}} v_i(k_i(\theta), \theta) \mu_{-i}(d\theta_{-i}|\theta_i) = \int_{\Theta_{-i}} v_i(\bar{k}_i(\theta_i), \theta_i) \mu_{-i}(d\theta_{-i}|\theta_i).$$

In sum, $\bar{k}_i$ is IPE to $k_i$. Moreover, since $\bar{k}$ is IPE to $k$, and $k$ is BIC, $\bar{k}$ is clearly BIC too.

"Only if". Assume for each $k$, some $\bar{k}$ is IPE to $k$. Thus, we must have (1). Similar to

---

28 A continuous function between locally compact Hausdorff spaces is closed.
"if" part we know its left hand side will be contained in an interval with two boundary points as $\int_{\Theta_{-i}} \inf_{k_i \in \mathcal{K}_i} v_i(k_i, \theta) \mu_{-i}(d\theta_{-i}|\theta_i)$ and $\int_{\Theta_{-i}} \sup_{k_i \in \mathcal{K}_i} v_i(k_i, \theta) \mu_{-i}(d\theta_{-i}|\theta_i)$, and the right hand side of (1) will be contained in an interval with two boundary points as $\inf_{k_i \in \mathcal{K}_i} \int_{\Theta_{-i}} v_i(k_i, \theta) \mu_{-i}(d\theta_{-i}|\theta_i)$ and $\inf_{k_i \in \mathcal{K}_i} \int_{\Theta_{-i}} v_i(k_i, \theta) \mu_{-i}(d\theta_{-i}|\theta_i)$. Thus, boundary preserving property must hold. □

Proof of Proposition 3.
Due to Lemma 1, we need to show that the boundary preserving property holds in a quasi-separable environment. For each $i$, we have

$$\int_{\Theta_{-i}} \sup_{k_i \in \mathcal{K}_i} [h_{ij}(k_{ij}, \theta_i)w_{ij}(\theta)] \mu_{-i}(d\theta_{-i}|\theta_i) = \int_{\Theta_{-i}} [\sup_{k_i \in \mathcal{K}_i} h_{ij}(k_{ij}, \theta_i)]w_{ij}(\theta) \mu_{-i}(d\theta_{-i}|\theta_i) = \sup_{k_i \in \mathcal{K}_i} [h_{ij}(k_{ij}, \theta_i)]\int_{\Theta_{-i}} w_{ij}(\theta) \mu_{-i}(d\theta_{-i}|\theta_i) = \sup_{k_i \in \mathcal{K}_i} \int_{\Theta_{-i}} h_{ij}(k_{ij}, \theta_i)w_{ij}(\theta) \mu_{-i}(d\theta_{-i}|\theta_i).$$

Thus, clearly,

$$\sup_{k_i \in \mathcal{K}_i} \int_{\Theta_{-i}} v_i(k_i, \theta) \mu_{-i}(d\theta_{-i}|\theta_i) = \int_{\Theta_{-i}} \sup_{k_i \in \mathcal{K}_i} v_i(k_i, \theta) \mu_{-i}(d\theta_{-i}|\theta_i).$$

In the similar argument,

$$\inf_{k_i \in \mathcal{K}_i} \int_{\Theta_{-i}} v_i(k_i, \theta) \mu_{-i}(d\theta_{-i}|\theta_i) = \int_{\Theta_{-i}} \inf_{k_i \in \mathcal{K}_i} v_i(k_i, \theta) \mu_{-i}(d\theta_{-i}|\theta_i).$$

Thus, by Lemma 1 we conclude that for any collective (respectively, collective BIC) mechanism $k$, there exists a bilateral (respectively, bilateral BIC) mechanism $\overline{k}$ IPE to $k$. The converse is straightforward. □

Proof of Proposition 4.
By Proposition 3, we can always find a bilateral BIC mechanism $\overline{k}$ interim-payoff-equivalent
to $k^*$. Hence,

$$\int_{\Theta} u(k^*(\theta), \theta) \mu(d\theta)$$

$$= \int_{\Theta} \left( \sum_{i=1}^{n} \sum_{j=1}^{m_i} (G_{ij}(h_{ij}(k_{ij}^*(\theta)w_{ij}(\theta), \theta_i) + L(\theta)) \mu(d\theta) \right)$$

$$= \int_{\Theta} \left( \sum_{i=1}^{n} \sum_{j=1}^{m_i} (G_{ij}(h_{ij}(k_{ij}^*(\theta)w_{ij}(\theta), \theta_i)) \mu(d\theta) + \int_{\Theta} L(\theta) \mu(d\theta) \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m_i} \left( \int_{\Theta} G_{ij}(h_{ij}(k_{ij}^*(\theta)w_{ij}(\theta), \theta_i)) \mu(d\theta) + \int_{\Theta} L(\theta) \mu(d\theta) \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m_i} \left( \int_{\Theta} \int_{\Theta_i} G_{ij}(h_{ij}(k_{ij}^*(\theta)w_{ij}(\theta), \theta_i)) \mu_{\Theta_i}(d\theta_i) + \int_{\Theta} L(\theta) \mu(d\theta) \right)$$

(By Jensen’s inequality.)

$$= \sum_{i=1}^{n} \sum_{j=1}^{m_i} \left( \int_{\Theta} \int_{\Theta_i} h_{ij}(k_{ij}^*(\theta)w_{ij}(\theta), \theta_i) \mu_{\Theta_i}(d\theta_i) + \int_{\Theta} L(\theta) \mu(d\theta) \right)$$

(By hypothesis (iii))

$$= \sum_{i=1}^{n} \sum_{j=1}^{m_i} \left( \int_{\Theta} \int_{\Theta_i} h_{ij}(k_{ij}^*(\theta)w_{ij}(\theta), \theta_i) \mu_{\Theta_i}(d\theta_i) + \int_{\Theta} L(\theta) \mu(d\theta) \right)$$

$$= \int_{\Theta} u(k^*(\theta), \theta) \mu(d\theta).$$

Since $k^*$ is the optimal solution to (P1), $\int_{\Theta} u(k^*(\theta), \theta) \mu(d\theta) = \int_{\Theta} u(k^*(\theta), \theta) \mu(d\theta)$. Thus, bilateral BIC mechanism $\overline{k}$ brings to PL the same expected (ex ante) payoff as $k^*$ does. Hence, P1 is strategically equivalent to P1’ and therefore P2. □

**Proof of Proposition 5.**

It is straightforward by the conditions under which the equality holds in the Jensen’s inequality. □

**Proof of Proposition 6.**

Clearly in quasi-linear environment, there exists some $(\overline{x}, \overline{t})$ IPE to $(\overline{x}, \overline{t})$. Thus, the
difference between $U^*$ and $U^{**}$

\[
U^* - U^{**} \leq \int_\Theta \left( \sum_{i=1}^n \sum_{j=1}^n g_i(o^{ij}, \theta) h_i(x_i^*(\theta), \theta_i) h_j(x_j^*(\theta), \theta_j) - \sum_{i=1}^n \sum_{j=1}^n g_{ij}(o^{ij}, \theta) h_i(x_i^*(\theta), \theta_i) h_j(x_j^*(\theta), \theta_j) \right) \mu(d\theta),
\]

for some $o^{ij}$ between $(h_i(x_i^*(\theta), \theta_i))_{i \in \mathcal{N}}$ and 0 and some $o^{ij}$ between $(h_i(x_i^*(\theta), \theta_i))_{i \in \mathcal{N}}$ and 0 for each $\theta$. The equality holds by Taylor expansion Theorem, hypothesis (iii), and Proposition 5.

Thus,

\[
U^* - U^{**} \leq \frac{1}{2} \int_\Theta \left\{ \sum_{i=1}^n \sum_{j=1}^n g_{ij}(o^{ij}, \theta) h_i(x_i^*(\theta), \theta_i) h_j(x_j^*(\theta), \theta_j) \right\} \mu(d\theta)
\]

\[
+ \sum_{i=1}^n \sum_{j=1}^n \left( g_{ij}(o^{ij}, \theta) h_i(x_i^*(\theta), \theta_i) h_i(x_i^*(\theta), \theta_i) \right) \mu(d\theta)
\]

\[
\leq \frac{1}{2} \int_\Theta \left\{ \sum_{i=1}^n \sum_{j=1}^n M |h_i(x_i^*(\theta), \theta_i) h_j(x_j^*(\theta), \theta_j)| + \sum_{i=1}^n \sum_{j=1}^n M |h_i(x_i^*(\theta), \theta_i) h_i(x_i^*(\theta), \theta_i)| \right\} \mu(d\theta)
\]

\[
= M \{ \sum_{i=1}^n \sum_{j=1}^n (\alpha_{ij} + \beta_{ij}) \}
\]

where $\alpha_{ij} = \int_\Theta |h_i(x_i^*(\theta), \theta_i) h_j(x_j^*(\theta), \theta_j)| \mu(d\theta)$, and $\beta_{ij} = \int_\Theta |h_i(x_i^*(\theta), \theta_i) h_j(x_j^*(\theta), \theta_j)| \mu(d\theta)$. $\alpha_{ij}$ and $\beta_{ij}$ will be nonnegative and finite, since each $x_i^*$ and $x_i^*$ are regular.

References


